105年 台灣大學 試題

解題老師：周易

系所：土木工程所(大地工程甲組、結構工程組、水利工程組)　科目：工程數學(A)

1. Find the characteristic polynomial, the eigenvalues and the eigenvectors of the following matrix: (20%)

$$A=\begin{bmatrix}2 & 1 & -1\\ 3 & 2 & -3\\ 3 & 1 & -2\end{bmatrix}$$

2. Prove the following vector theorem: (20%)

if $\vec{\nabla}\cdot\vec{u}=0$

and $\vec{\nabla}\times\vec{u}=\vec{\omega}$

then $\nabla^2\vec{u}=\vec{\nabla}\cdot\vec{\nabla}\vec{u}=-\vec{\omega}$ and write out in Cartesian components if

$\vec{u}=u_1\vec{i}+u_2\vec{j}+u_3\vec{k}$ and $\vec{\omega}=\omega_1\vec{i}+\omega_2\vec{j}+\omega_3\vec{k}$

Explain the physical meanings in fluid mechanics.

(Hint: use the vector identity of vector triple products

$$\vec{A}\times(\vec{B}\times\vec{C})=(\vec{A}\cdot\vec{C})\vec{B}-(\vec{A}\cdot\vec{B})\vec{C}\qquad)$$

3. Solve the following initial value problem by the Laplace transform method; plot the input, output graphs and also interpret the physical meanings in hydrology: (20%)

$y''+3y'+2y=\delta(t-1)-\delta(t-2)$

$y(0)=y'(0)=0$

where δ is the Dirac delta function.

4. Find the Fourier series expansion of the following period function $f(x)$: (20%)

$f(x)=x-2L \quad 2L\le x\le 3L$

$\quad=x-4L \quad 3L\le x\le 4L$

and $f(x+2L)=f(x)$.

5. Solve the following partial differential equation in a square cavity by using separation of variables and Fourier series expansion method

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \qquad 0 \le x \le 1 \text{ and } 0 \le y \le 1$$

(1) Subject to the boundary conditions (BCs): (10%)

$u(x,0) = u(0,y) = u(1,y) = 0$ and $u(x,1) = 1$

what is the name of this equation, and where it appears in engineering?

(2) Subject to the BCs: $u(0,y) = u(x,0) = 0$ and $u(x,1) = u(1,y) = 1$. (10%)

(Hint: use the superposition principle)

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1. $A\vec{x}=\lambda\vec{x}$, $(A-\lambda I)\vec{x}=\vec{0}$

$$\begin{bmatrix}2-\lambda & 1 & -1\\ 3 & 2-\lambda & -3\\ 3 & 1 & -2-\lambda\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\vec{0},\quad |A-\lambda I|=-(\lambda-1)(\lambda+1)(\lambda-2)$$

上述爲特徵多項式, 特徵值 $\lambda=-1,\ 1,\ 2$

當 $\lambda_1=-1$　　eigenvectors

$$\begin{bmatrix}3 & 1 & -1\\ 3 & 3 & -3\\ 3 & 1 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\vec{0},\qquad \vec{x}=c_1\begin{bmatrix}0\\ 1\\ 1\end{bmatrix}$$

當 $\lambda_2=1$

$$\begin{bmatrix}1 & 1 & -1\\ 3 & 1 & -3\\ 3 & 1 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\vec{0},\qquad \vec{x}=c_2\begin{bmatrix}1\\ 0\\ 1\end{bmatrix}$$

當 $\lambda_3=2$

$$\begin{bmatrix}0 & 1 & -1\\ 3 & 0 & -3\\ 3 & 1 & -4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\vec{0},\qquad \vec{x}=c_3\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}$$

2. $\because\ \nabla\times(\nabla\times\vec{u})=\nabla(\nabla\cdot\vec{u})-(\nabla\cdot\nabla)\vec{u}$

代入題目條件 $\nabla\cdot\vec{u}=0$, $\nabla\times\vec{u}=\vec{\omega}$

得 $\nabla\times\vec{\omega}=\nabla 0-\nabla^2\vec{u}$, $\nabla^2\vec{u}=-\nabla\times\vec{\omega}$, 題目給 $\nabla^2\vec{u}=-\vec{\omega}$ 有誤

$$\nabla^2(u_1\vec{i}+u_2\vec{j}+u_3\vec{k})=-\left(\frac{\partial}{\partial x}\vec{i}+\frac{\partial}{\partial y}\vec{j}+\frac{\partial}{\partial z}\vec{k}\right)\times(\omega_1\vec{i}+\omega_2\vec{j}+\omega_3\vec{k})$$

$$\nabla^2u_1=-\left(\frac{\partial\omega_3}{\partial y}-\frac{\partial\omega_2}{\partial z}\right),\quad \nabla^2u_2=-\left(\frac{\partial\omega_1}{\partial z}-\frac{\partial\omega_3}{\partial x}\right),\quad \nabla^2u_3=-\left(\frac{\partial\omega_2}{\partial x}-\frac{\partial\omega_1}{\partial y}\right)$$

其中 $\nabla^2=\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}+\dfrac{\partial^2}{\partial z^2}$

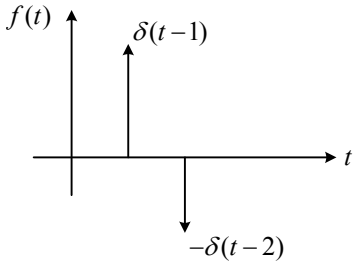
流體力學物理涵意, 請參閱流體力學相關書籍

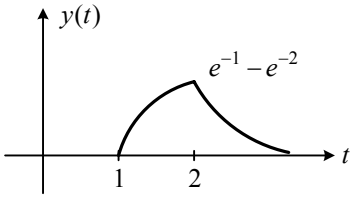
3. $L[y''+3y'+2y]=L[\delta(t-1)-\delta(t-2)]$，$s^2Y+3sY+2Y=e^{-s}-e^{-2s}$

$$Y=\frac{1}{s^2+3s+2}(e^{-s}-e^{-2s})=\left(\frac{1}{s+1}-\frac{1}{s+2}\right)\cdot(e^{-s}-e^{-2s})$$

$$y=L^{-1}\left[\left(\frac{1}{s+1}-\frac{1}{s+2}\right)e^{-s}-\left(\frac{1}{s+1}-\frac{1}{s+2}\right)e^{-2s}\right]$$

$$y=u(t-1)\cdot[e^{-(t-1)}-e^{-2(t-1)}]-u(t-2)\cdot[e^{-(t-2)}-e^{-2(t-2)}]$$





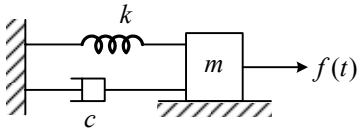
當 $0<t<1$，$y(t)=0$，當 $1<t<2$，$y(t)=e^{-(t-1)}-e^{-2(t-1)}$

$y(1^+)=0$，$y(2^-)=e^{-1}-e^{-2}$

當 $t>2$，$y(t)=[e^{-(t-1)}-e^{-2(t-1)}]-[e^{-(t-2)}-e^{-2(t-2)}]$

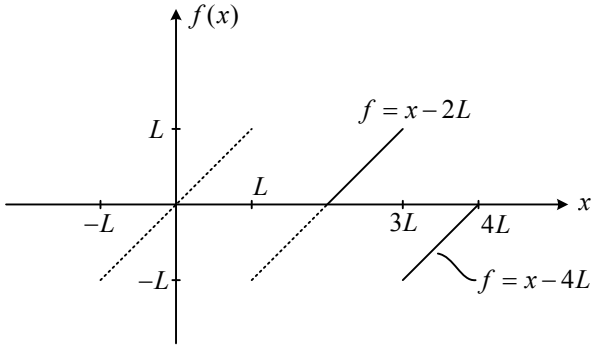
$y(2^+)=(e^{-1}-e^{-2})-0$，$y(\infty)=0$

physical meaning: 在 MCK 系統中，$m=1$，$c=3$，$k=2$, input $f(t)$



在 $t=1$ sec, 施加一單位正衡量，$t=2$ sec, 施加負一單位衡量

4.



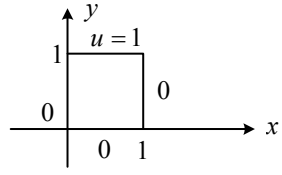
$f(x)$ 爲週期 $2L$ 奇函數, 取 Fourier sine series

$$f(x)=\sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L},\quad b_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

$$b_n=\frac{2}{L}\int_0^L x\sin\frac{n\pi x}{L}dx=\frac{2}{L}\cdot\frac{-L}{n\pi}x\cos\frac{n\pi x}{L}\Big|_0^L=\frac{-2L}{n\pi}(-1)^n$$

$$f(x)=\sum_{n=1}^{\infty}\frac{2L}{n\pi}(-1)^{n+1}\sin\frac{n\pi x}{L}$$

5. (1) 令 $u(x,y)=X(x)\cdot Y(y)$ 代入 $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$, $X''Y+XY''=0$



$\dfrac{X''}{X}=-\dfrac{Y''}{Y}=\lambda$, $X''-\lambda X=0$, $Y''+\lambda Y=0$

$X(0)=0$, $X(1)=0$, $Y(0)=0$

當 $\lambda>0$, 令 $\lambda=\beta^2$, $\beta>0$, $X=c_1\cosh\beta x+c_2\sinh\beta x$

代入 $X(0)=0$, $c_1=0$, 代入 $X(1)=0$, $c_2=0$

當 $\lambda=0$, $X=c_1+c_2x$, 代入 $X(0)=0$, $c_1=0$, $X(1)=0$, $c_2=0$

當 $\lambda<0$, 令 $\lambda=-\beta^2$, $\beta>0$, $X=c_1\cos\beta x+c_2\sin\beta x$

代入 $X(0)=0$, $c_1=0$, $X=c_2\sin\beta x$

代入 $X(1)=0$, $c_2\sin\beta=0$

當 $\beta=n\pi$, $n=1,2,3,\cdots$, c_2 爲任意數

特徵值 $\lambda_n=-\beta^2=-n^2\pi^2$, 特徵函數 $X_n(x)=c_n\sin n\pi x$

將 $\lambda_n=-n^2\pi^2$ 代入 $Y''+\lambda Y=0$, $Y''-n^2\pi^2Y=0$, $Y=k_1\cosh n\pi y+k_2\sinh n\pi y$

代入 $Y(0)=0$, $k_1=0$, $Y=k_2\cdot\sinh n\pi y$

PDE 之解 $u=X(x)\cdot Y(y)=c_n\sin n\pi x\cdot k_2\sinh n\pi y$, 令 $B_n=c_n\cdot k_2$

$u(x,y)=B_n\cdot\sin n\pi x\cdot\sinh n\pi y$, $n=1,2,3,\cdots$

都是齊性 PDE $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$ 之齊性解

齊性 PDE 通解 $u(x,y)=\sum_{n=1}^{\infty}B_n\cdot\sinh n\pi y\cdot\sin n\pi x$

代入條件 $u(x,1)=1=\sum_{n=1}^{\infty}B_n\cdot\sinh n\pi\cdot\sin n\pi x$

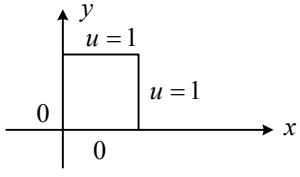
$$B_n \cdot \sinh n\pi = \frac{\int_0^1 1\sin n\pi x dx}{\int_0^1 \sin^2 n\pi x dx},\quad B_n = \frac{2}{n\pi}[1-(-1)^n]\cdot\frac{1}{\sinh n\pi}$$

$u(x,y) = \sum_{n=1,3,5}^{\infty} \frac{4}{n\pi}\frac{\sinh n\pi y}{\sinh n\pi}\cdot\sin n\pi x$ 爲 PDE 之解

本題之 PDE 稱爲 Laplace equation, 爲穩定狀態下,

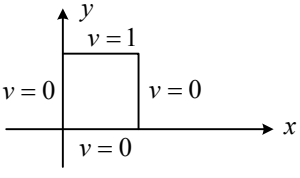
平板溫度(or 靜電位)分佈

(2) $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$

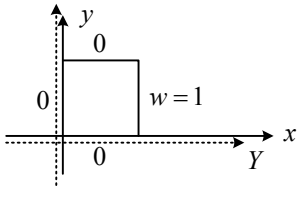


利用重疊原理, 令 $u(x,y)=v(x,y)+w(x,y)$

$v(x,y)$ 滿足 $\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$



$w(x,y)$ 滿足 $\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=0$



由(1)知, $v(x,y)=\sum_{n=1,3,5}^{\infty}\frac{4}{n\pi}\frac{\sinh n\pi y}{\sinh n\pi}\cdot\sin n\pi x$

利用座標轉換解 $w(x,y)$

$$w(x,y)=v(X,Y)=v(y,x)=\sum_{n=1,3,5}^{\infty}\frac{4}{n\pi}\frac{\sinh n\pi x}{\sinh n\pi}\sin n\pi y$$

$$u(x,y)=v(x,y)+w(x,y)$$

$$u(x,y)=\sum_{n=1,3,5}^{\infty}\frac{4}{n\pi}\cdot\frac{1}{\sinh n\pi}(\sinh n\pi y\cdot\sin n\pi x+\sinh n\pi x\cdot\sin n\pi y)$$