105年 台灣大學 試題

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科目：工程數學(C)

1. A semi-infinite plate coincides with the region defined by $0 \le x \le \pi$, $y \ge 0$. The left end is at temperature, $u(0, y) = e^{-y}$, and the right end is held at temperature zero. The bottom of the plate is insulated. Please solve the boundary-value problem for the stead-state temperature $u(x, y)$. (10%)

$$u(x,y) = \frac{2}{\pi}\int_0^{\infty} \frac{\sinh[s(\alpha)]}{[t(\alpha)\sinh[r(\alpha)]]}\cos(\alpha y)d\alpha .$$

(A) $r(\alpha) = \alpha x$ (B) $s(\alpha) = \alpha(\pi - y)$
(C) $s(\alpha) = \alpha(\pi - x)$ (D) $t(\alpha) = 1 + \alpha^2$
(E) $t(\alpha) = 1 + \alpha x$

2. Use the result $\int_{-\infty}^{\infty} \exp(-x^2)dx = \sqrt{\pi}$ to find the Fourier integral transform

$$F(\alpha) = \int_{-\infty}^{\infty} \exp[-x^2/(4p^2)]\exp(i\alpha x)dx = s(p)\sqrt{\pi}\exp[-t(p)\alpha^2] \quad (10\%)$$

(A) $s(p) = -2p$ (B) $s(p) = p^2$ (C) $s(p) = 2p$ (D) $t(p) = 4p^2$
(E) $t(p) = p^2$

3. The vertical displacement $u(x,t)$ of an infinitely long string is determined from the initial-value problem $a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$. The string is released from rest from the initial displacement $f(x) = \exp(-x^2)$. (5%)

$$u(x,t) = \exp[-x^2 + s(t)]\cosh[r(t)x]$$

(A) $s(t) = -a^2t^2$ (B) $s(t) = -2at$ (C) $r(t) = -2at$ (D) $r(t) = at$
(E) $r(t) = 2at$

4. Consider a dynamic system with a time-varying state $x(t)$ which satisfies the following differential equation:

$$(t^2 - 3)x'' + 4tx' + 2x = 0, \quad t \ge 0 .$$

Under the conditions $x(0) \in (1,3)$ and $x'(0) \in (-1,1)$, find all possible values of $x(3)$ from below. (10%)

(A) -4 (B) -2 (C) 0 (D) $1/2$ (E) 1

Hint: Use power series to solve $x(t)$ and simplify your solution

5. Solve the following initial value problem: $y(0)=1$, $y'(0)=3$, and

$$y''-y=g(x)=\begin{cases}0 & x<0\\ x^3 & x\ge 0\end{cases}.$$

Choose from below the value of $(y(-1), y(1))$. (10%)

(A) $(5e-4e^{-1}-7,\ 5e^{-1}-4e+7)$ (B) $(5e-4e^{-1}-7,\ 2e^{-1}-e)$

(C) $(2e-e^{-1},\ 5e^{-1}-4e+7)$ (D) $(2e-e^{-1},\ 2e^{-1}-e)$

(E) None of the above

6. A function $y(x)$ satisfies $(y')^2=9x^4y$, $y(0)=0$. Find all possible values of $y'(-2)$ from below. (5%)

(A) -16 (B) 0 (C) 24 (D) -32 (E) -48

7. Use Gaussian elimination procedures to find the nullity of the matrix below: (10%)

$$\begin{bmatrix}1 & 0 & -2 & -1 & 0 & -1\\ 2 & -1 & -6 & -2 & 0 & -4\\ 0 & 1 & 2 & 1 & 1 & 1\\ -1 & 2 & 6 & 3 & 1 & 2\end{bmatrix}$$

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4 (F) 5

8. Suppose that $T: R^3\to R^3$ is the reflection of R^3 about the x-y plane. Which of the following statements are true (multiple choice): (15%)

(A) The null space of T is $\left\{\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$

(B) T is one-to-one

(C) T is onto,

(D) The standard matrix of T is $\begin{bmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}$

9. Given the matrix below (10%)

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

which of the following statements is (are) true? (10%)

(A) The columns form an orthonormal basis for the plane R^3.

(B) The rows are orthonormal.

(C) The inverse of this matrix is $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(D) The inverse of this matrix is its transpose.

(E) None of the above.

10. For the matrix below (10%)

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

which of the following statements is (are) true?

(A) 1 is its eigenvalue.

(B) -1 is its eigenvalue.

(C) It has only real eigenvalue(s).

(D) It has complex eigenvalue(s).

(E) None of the above.

11. The eigenvectors of matrix A are put into the columns of the eigenvector matrix S. If the eigenvectors of A are linearly independent, then which of the following statements is (are) true?

(A) S is invertible.

(B) S is diagonalizable.

(C) A is invertible.

(D) A is diagonalizable.

(E) None of the above.

105年 台灣大學 解答

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1. $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$， $0<x<\pi$， $0<y<\infty$

$u(0,y)=e^{-y}$， $u(\pi,y)=0$， $u_y(x,0)=0$

取 y 方向 Fourier cosine 積分

$u=\frac{2}{\pi}\int_0^\infty A\cos\alpha y d\alpha$， $A=\int_0^\infty u(x,y)\cos\alpha y dy$

$\int_0^\infty \frac{\partial^2 u}{\partial x^2}\cos\alpha y dy+\int_0^\infty \frac{\partial^2 u}{\partial y^2}\cos\alpha y dy=0$，部份積分二次

$\frac{d^2}{dx^2}A-\alpha^2 A=0$， $A=c_1\cosh\alpha(x-\pi)+c_2\sinh\alpha(x-\pi)$

又 $A=\int_0^\infty u(x,y)\cos\alpha y dy$

取 $x=\pi$， $u(\pi,y)=0$， $A=0$， $c_1=0$

$x=0$， $u(0,y)=e^{-y}$， $A=\int_0^\infty e^{-y}\cos\alpha y dy=\frac{1}{1+\alpha^2}$

$c_2\cdot\sinh(-\alpha\pi)=\frac{1}{1+\alpha^2}$， $c_2=\frac{-1}{1+\alpha^2}\cdot\frac{1}{\sinh\alpha\pi}$

$A=\frac{-1}{1+\alpha^2}\cdot\frac{1}{\sinh\alpha\pi}\sinh\alpha(x-\pi)=\frac{1}{1+\alpha^2}\cdot\frac{\sinh\alpha(\pi-x)}{\sinh\alpha\pi}$

代入 $u=\frac{2}{\pi}\int_0^\infty A\cos\alpha y d\alpha$

$u=\frac{2}{\pi}\int_0^\infty \frac{1}{1+\alpha^2}\cdot\frac{\sinh\alpha(\pi-x)}{\sinh\alpha\pi}\cos\alpha y d\alpha$

又題目給 $u(x,y)=\frac{2}{\pi}\int_0^\infty \frac{1}{t(\alpha)}\cdot\frac{\sinh[s(\alpha)]}{\sinh[r(\alpha)]}\cos\alpha y\, d\alpha$

得 $t(\alpha)=1+\alpha^2$， $r(\alpha)=\alpha\pi$， $s(\alpha)=\alpha(\pi-x)$

本題取(C), (D)

2. $\because F[e^{-ax^2}]=\int_{-\infty}^\infty e^{-ax^2}e^{-iwx}dx=\sqrt{\frac{\pi}{a}}\cdot e^{-\frac{w^2}{4a}}$

取共軛 $\int_{-\infty}^{\infty} e^{-ax^2} e^{iwx} dx = \sqrt{\frac{\pi}{a}} e^{-\frac{w^2}{4a}}$

$\int_{-\infty}^{\infty} e^{-\frac{x^2}{4p^2}} e^{i\alpha x} dx = \sqrt{4p^2\pi} e^{-p^2\alpha^2} = 2p\sqrt{\pi} e^{-p^2\alpha^2} = s(p)\cdot\sqrt{\pi}\exp[-t(p)\alpha^2]$

得 $s(p) = 2p$， $t(p) = p^2$，本題取(C), (E)

3. $a^2 \cdot F\left[\frac{\partial^2 u}{\partial x^2}\right] = F\left[\frac{\partial^2 u}{\partial t^2}\right]$，自行假設 $u(\pm\infty, t) = 0$

$-a^2w^2U = \frac{d^2}{dt^2}U$， $(D_t{}^2 + a^2w^2)U = 0$， $U = c_1\cos awt + c_2\sin awt$

又 $U = F[u(x,t)] = \int_{-\infty}^{\infty} u(x,t)e^{-iwx} dx$

題目條件 $u_t(x,0) = 0$， $u(x,0) = f(x) = e^{-x^2}$

取 $\left.\frac{dU}{dt}\right|_{t=0} = \int_{-\infty}^{\infty} u_t(x,0)e^{-iwx} dx = \int_{-\infty}^{\infty} 0e^{-iwx} dx = 0$，

得 $awc_2 = 0$， $c_2 = 0$

取 $t = 0$， $c_1 = \int_{-\infty}^{\infty} u(x,0)e^{-iwx} dx = \int_{-\infty}^{\infty} e^{-x^2} e^{-iwx} dx = \sqrt{\pi} e^{-\frac{w^2}{4}}$

$U = \sqrt{\pi} e^{-\frac{1}{4}w^2}\cos awt$， $u = F^{-1}[U] = \frac{1}{2\pi}\int_{-\infty}^{\infty} Ue^{iwx} dw$

$u = \frac{1}{2\pi}\int_{-\infty}^{\infty} \sqrt{\pi} e^{-\frac{1}{4}w^2}\cos awt\cdot e^{iwx} dw$

$= \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty} \frac{1}{2}(e^{iawt} + e^{-iawt})e^{iwx}\cdot e^{-\frac{1}{4}w^2} dw$

$= \frac{1}{4\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{4}w^2}[e^{iw(x-at)} + e^{iw(x+at)}]dw$

$\because \int_{-\infty}^{\infty} e^{-ax^2} e^{iwx} dx = \sqrt{\frac{\pi}{a}} e^{-\frac{w^2}{4a}}$

$u(x,t) = \frac{1}{4\sqrt{\pi}}[\sqrt{4\pi} e^{-(x-at)^2} + \sqrt{4\pi} e^{-(x+at)^2}]$

$= \frac{1}{2}[e^{-(x^2-2axt+a^2t^2)} + e^{-(x^2+2axt+a^2t^2)}]$

$u = \frac{1}{2} e^{-(x^2+a^2t^2)}\cdot[e^{2axt} + e^{-2axt}] = \exp(-x^2 - a^2t^2)\cdot\cosh 2axt$

又 $u = \exp[-x^2 + s(t)]\cdot\cosh[r(t)x]$，得 $s(t) = -a^2t^2$， $r(t) = 2at$

本題取(A), (E)

4. $(t^2-3)x''+4tx'+2x=0$，令 $x(0)=c_1$，$x'(0)=c_2$

$x(t)=x(0)+x'(0)t+\frac{1}{2}x''(0)\cdot t^2+\frac{1}{3!}x'''(0)t^3+\cdots$，$|t-0|<\sqrt{3}$，$-\sqrt{3}<t<\sqrt{3}$

$t=3$，代入級中，發散，無法算出 $x(3)$

對奇異點 $t=\sqrt{3}$，取 Frobenius series

$x(t)=(t-\sqrt{3})^r\sum_{n=0}^{\infty}c_n(t-\sqrt{3})^n$，$0<\left|t-\sqrt{3}\right|<\infty$，令 $z=t-\sqrt{3}$，$0<|z|<\infty$

$x(z)=z^r\sum_{n=0}^{\infty}c_n\cdot z^n$ 代入 ODE，$x(z)=\sum_{n=0}^{\infty}c_nz^{n+r}$

$[(z+\sqrt{3})^2-3]x''(z)+4(z+\sqrt{3})x'(z)+2x(z)=0$

$(z^2+2\sqrt{3}z)x''(z)+4z\cdot x'(z)+4\sqrt{3}\cdot x'(z)+2x(z)=0$

$z^2x''+2\sqrt{3}z\cdot x''+4z\cdot x'+4\sqrt{3}x'+2x=0$

$\sum_{n=0}^{\infty}(n+r)(n+r-1)c_nz^{n+r}+\sum_{n=0}^{\infty}2\sqrt{3}\cdot(n+r)(n+r-1)c_n\cdot z^{n+r-1}$

$+\sum_{n=0}^{\infty}4(n+r)c_nz^{n+r}+\sum_{n=0}^{\infty}4\sqrt{3}(n+r)c_nz^{n+r-1}+\sum_{n=0}^{\infty}2c_nz^{n+r-1}=0$

$\sum_{n=0}^{\infty}(n+r)(n+r-1+4)c_n\cdot z^{n+r}$

$+\sum_{n=0}^{\infty}[2\sqrt{3}(n+r)^2-2\sqrt{3}(n+r)+4\sqrt{3}(n+r)+2]c_nz^{n+r-1}=0$

可算出指標方程式二根 r_1, r_2

$x=c_1(t-\sqrt{3})^{r_1}[1+\square(t-\sqrt{3})+\square(t-\sqrt{3})^2+\cdots]$

$+c_2(t-\sqrt{3})^{r_2}[1+\square(t-\sqrt{3})+\square(t-\sqrt{3})^2+\cdots]$

代入 $x(0)$，$x'(0)$ 不易找出 c_1，c_2

結論：本題用題目所給之提示，無法算出

另解：$(t^2-3)x''+4tx'+2x=0$，這是正合 ODE

$[(t^2-3)\cdot x']'-2tx'+4tx'+2x=0$，$[(t^2-3)x']'+2tx'+2x=0$

$[(t^2-3)x']'+(2tx)'=0$，積分一次，$(t^2-3)x'+2tx=c_1$

代入 $t=0$，$-3x'(0)=c_1$，得 $(t^2-3)\frac{dx}{dt}+2tx=c_1$

$(t^2-3)dx+2txdt=c_1dt$，$(t^2-3)dx+xd(t^2-3)=c_1dt$

$(t^2-3)x=c_1t+c_2$，代入 $t=0$，$-3x(0)=c_2$

$x=\dfrac{c_1}{t^2-3}t+\dfrac{c_2}{t^2-3}$，$x(t)=\dfrac{t}{t^2-3}(-3x'(0))+\dfrac{-3}{t^2-3}x(0)$

$x(3)=\dfrac{3}{6}(-3)\cdot x'(0)-\dfrac{3}{6}x(0)=\dfrac{-3}{2}x'(0)-\dfrac{1}{2}x(0)$

$\because\ 1<x(0)<3$，$-1<x'(0)<1$

max. $x(3)=\dfrac{-3}{2}\cdot(-1)-\dfrac{1}{2}\cdot 1=1$

min. $x(3)=\dfrac{-3}{2}\cdot 1-\dfrac{1}{2}\cdot 3=-3$，得 $-3<x(3)<1$

本題取(B), (C), (D)

5. $g(x)=u(x)\cdot x^3$，$L[y''-y]=L[g(x)]$

$[s^2Y-sy(0)-y'(0)]-Y=e^{-0s}\cdot L[x^3]$

$(s^2-1)Y-s-3=e^{-0s}\cdot\dfrac{6}{s^4}$，$Y=\dfrac{s+3}{s^2-1}+\dfrac{6}{s^4(s^2-1)}e^{-0s}$

$Y=\dfrac{2}{s-1}+\dfrac{-1}{s+1}+\left(\dfrac{-6}{s^2}+\dfrac{-6}{s^4}+\dfrac{3}{s-1}+\dfrac{-3}{s+1}\right)e^{-0s}$

$y=L^{-1}[Y]=2e^x-e^{-x}+u(x)\cdot[-6x-x^3+3e^x-3e^{-x}]$

$y(-1)=2e^{-1}-e^1$，$y(1)=2e-e^{-1}+(-6-1+3e-3e^{-1})$

$y(1)=5e-4e^{-1}-7$

$(y(-1),\ y(1))=(2e^{-1}-e,\ 5e-4e^{-1}-7)$，本題取(E)

6. $y'=\pm 3x^2\sqrt{y}$，$\dfrac{dy}{\sqrt{y}}=\pm 3x^2dx$，$2\sqrt{y}=\pm x^3+c$，$y=\left[\pm\left(\dfrac{1}{2}x^3+\dfrac{c}{2}\right)\right]^2$

$y=\left(\dfrac{1}{2}x^3+c_1\right)^2$　代入 $y(0)=0$，$c_1=0$，$y=\dfrac{1}{4}x^6$

$y'=\dfrac{3}{2}x^5$，$y'(-2)=\dfrac{3}{2}(-2)^5=-48$

$\because\ y(x)=0$ 滿足 ODE, 也滿足條件 $y(0)=0$，也是解, 得 $y'(-2)=0$

本題取(B), (E)

7.

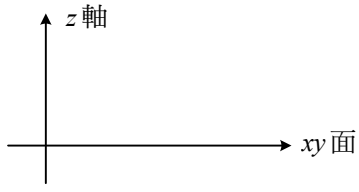
$$A=\begin{bmatrix}1&0&-2&-1&0&-1\\2&-1&-6&-2&0&-4\\0&1&2&1&1&1\\-1&2&6&3&1&2\end{bmatrix}\xrightarrow[R_{14}(1)]{R_{12}(-2)}\begin{bmatrix}1&0&-2&-1&0&-1\\0&-1&-2&0&0&-2\\0&1&2&1&1&1\\0&2&4&2&1&1\end{bmatrix}$$

$$\xrightarrow[R_{24}(2)]{R_{23}(1)}\begin{bmatrix}1&0&-2&-1&0&-1\\0&-1&-2&0&0&-2\\0&0&0&1&1&-1\\0&0&0&2&1&-3\end{bmatrix}\xrightarrow[R_{34}(-2)]{R_{2}(-1)}\begin{bmatrix}1&0&-2&-1&0&-1\\0&1&2&0&0&2\\0&0&0&1&1&-1\\0&0&0&0&-1&-1\end{bmatrix}$$

$rank(A)=4$，又 $rank(A)+nullity(A)=6$，$nullity(A)=2$

本題取(C)

8.



$$T\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}1\\0\\0\end{bmatrix},\quad T\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}0\\1\\0\end{bmatrix},\quad T\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}0\\0\\-1\end{bmatrix},\quad \text{得 } T=\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}$$

$T=T^{-1}=T^{t}$，T 可逆，T 爲一對一且 omto，$N(T)=\{\vec{0}\}$

本題取(B), (C)

9.

$$A=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix},\quad A^{t}=A^{-1}=\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}$$

A 由歸一正交向量，當 $col(A)$，$row(A)$ 基底

A 之行向量形成 R^3 之歸一正交基底，但是 R^3 不是 plane

本題取(B), (D)

10.

$$|A-\lambda I| = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{vmatrix} = -\lambda^3 + 1 = -(\lambda - 1)(\lambda^2 + \lambda + 1)$$

$\lambda = 1,\ \frac{1}{2}(-1 \pm \sqrt{3}i)$

本題取(A), (D)

11. A 之特徵向量都線性獨立，A 可對角化 $A = SDS^{-1}$

但是 A 不一定可逆。S 可逆，S 不一定可對角化

本題取(A), (D)