105年 台灣大學 試題

解題老師：周易

系所：工程科學及海洋工程所(甲、乙組)　科目：工程數學(F)

1. Find a family of curves which is the orthogonal trajectories of the curves: (10%)
$x^2+(y-c)^2=c^2$, where c is an arbitrary constant.

2. The 2-D Laplacian operator in Cartesian coordinate is: (10%)
$$\nabla^2=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right).$$
Transform this operator in Polar coordinate (r,θ) with $x=r\cos\theta$; $y=r\sin\theta$

3. Find the complete solution of the following nonhomogeneous matrix equation: (20%)
$$x_1'=x_1-x_2+\frac{e^{-t}}{1+t^2}$$
$$x_2'=2x_1-2x_2+\frac{2e^{-t}}{1+t^2}$$

4. Find the inverse Laplace transform of the function $Y(s)=\dfrac{2s^2-s}{(s^2+4)^2}$ (20%)

5. Consider the following eigenvalues problem of $y(x)$: (20%)
$y''+\lambda^2 y=0 \quad -p\ge x\le p$.
B.C. $\begin{cases}y(-p)=y(p)\\ y'(-p)=y'(p)\end{cases}$
if λ_m and λ_n are two distinct eigenvalues of the problem, show that the corresponding eigenfunctions $y_m(x)$ and $y_n(x)$ are orthogonal in $(-p,p)$.

6. What is the Fourier expansion of the periodic function (20%)
$f(t)=4-t^2 \quad -2\le t\le 2$
And also prove that $1-\dfrac{1}{2^2}+\dfrac{1}{3^2}-\dfrac{1}{4^2}+\dfrac{1}{5^2}-\cdots=\dfrac{\pi^2}{12}$

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1. 令曲線 u 爲 $x^2+(y-c)^2=c^2$，且曲線 v 與它正交

$dx^2+d(y-c)^2=dc^2$，$2xdx+2(y-c)dy=0$，$\dfrac{dy}{dx}=-\dfrac{x}{y-c}$

又 $x^2+y^2-2yc+c^2=c^2$，$c=\dfrac{x^2+y^2}{2y}$，$y-c=y-\dfrac{x^2+y^2}{2y}=\dfrac{y^2-x^2}{2y}$

得 $\left.\dfrac{dy}{dx}\right|_u=-\dfrac{2xy}{y^2-x^2}$，代入 $\left.\dfrac{dy}{dx}\right|_u\cdot\left.\dfrac{dy}{dx}\right|_v=-1$

$-\dfrac{2xy}{y^2-x^2}\cdot\left.\dfrac{dy}{dx}\right|_v=-1$，$2xydy=(y^2-x^2)dx$，$2xydy-y^2dx=-x^2dx$

$y(2xdy-ydx)=-x^2dx$，$\dfrac{yd(y^2x^{-1})}{yx^{-2}}=-x^2dx$，$x^2d\left(\dfrac{y^2}{x}\right)=-x^2dx$

$\dfrac{y^2}{x}=-x+\alpha$，$y^2+x^2-\alpha x=0$，$y^2+\left(x-\dfrac{1}{2}\alpha\right)^2=\left(\dfrac{1}{2}\alpha\right)^2$

令新 $\alpha=\dfrac{1}{2}\alpha$，$(x-\alpha)^2+y^2=\alpha^2$ 與 $x^2+(y-c)^2=c^2$ 正交

2. $\dfrac{\partial^2u}{\partial x^2}+\dfrac{\partial^2u}{\partial y^2}=0$，令 $u=u(r,\theta)$，$r=\sqrt{x^2+y^2}$，$\theta=\tan^{-1}\dfrac{y}{x}$

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial u}{\partial\theta}\frac{\partial\theta}{\partial x}=\frac{\partial u}{\partial r}\frac{x}{\sqrt{x^2+y^2}}+\frac{\partial u}{\partial\theta}\cdot\frac{-y}{x^2+y^2}$$

$$\therefore\ \frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\cos\theta-\frac{\partial u}{\partial\theta}\frac{\sin\theta}{r}$$

$$\frac{\partial^2u}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\cos\theta-\frac{\partial u}{\partial\theta}\frac{\sin\theta}{r}\right)$$

$$=\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\cos\theta-\frac{\partial u}{\partial\theta}\frac{\sin\theta}{r}\right)\frac{\partial r}{\partial x}+\frac{\partial}{\partial\theta}\left(\frac{\partial u}{\partial r}\cos\theta-\frac{\partial u}{\partial\theta}\frac{\sin\theta}{r}\right)\frac{\partial\theta}{\partial x}$$

$$\frac{\partial^2 u}{\partial x^2}=\left(\frac{\partial^2 u}{\partial r^2}\cos\theta-\frac{\partial^2 u}{\partial r\partial\theta}\frac{\sin\theta}{r}+\frac{\partial u}{\partial\theta}\frac{\sin\theta}{r^2}\right)\cos\theta$$
$$+\left(\frac{\partial^2 u}{\partial r\partial\theta}\cos\theta-\frac{\partial u}{\partial r}\sin\theta-\frac{\partial^2 u}{\partial\theta^2}\frac{\sin\theta}{r}-\frac{\partial u}{\partial\theta}\frac{\cos\theta}{r}\right)\left(-\frac{\sin\theta}{r}\right)$$
$$=\frac{\partial^2 u}{\partial r^2}\cos^2\theta-2\frac{\partial^2 u}{\partial r\partial\theta}\frac{\cos\theta\sin\theta}{r}+2\frac{\partial u}{\partial\theta}\frac{\sin\theta\cos\theta}{r^2}$$
$$+\frac{\partial u}{\partial r}\frac{\sin^2\theta}{r}+\frac{\partial^2 u}{\partial\theta^2}\frac{\sin^2\theta}{r^2}$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial u}{\partial\theta}\frac{\partial\theta}{\partial y}=\frac{\partial u}{\partial r}\frac{y}{\sqrt{x^2+y^2}}+\frac{\partial u}{\partial\theta}\cdot\frac{x}{x^2+y^2}$$
$$=\frac{\partial u}{\partial r}\sin\theta+\frac{\partial u}{\partial\theta}\frac{\cos\theta}{r}$$

$$\frac{\partial^2 u}{\partial y^2}=\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\sin\theta+\frac{\partial u}{\partial\theta}\frac{\cos\theta}{r}\right)$$
$$=\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\sin\theta+\frac{\partial u}{\partial\theta}\frac{\cos\theta}{r}\right)\frac{\partial r}{\partial y}+\frac{\partial}{\partial\theta}\left(\frac{\partial u}{\partial r}\sin\theta+\frac{\partial u}{\partial\theta}\frac{\cos\theta}{r}\right)\frac{\partial\theta}{\partial y}$$
$$=\left(\frac{\partial^2 u}{\partial r^2}\sin\theta+\frac{\partial^2 u}{\partial r\partial\theta}\frac{\cos\theta}{r}-\frac{\partial u}{\partial\theta}\frac{\cos\theta}{r^2}\right)\sin\theta$$
$$+\left(\frac{\partial^2 u}{\partial r\partial\theta}\sin\theta+\frac{\partial u}{\partial r}\cos\theta-\frac{\partial^2 u}{\partial\theta^2}\frac{\cos\theta}{r}-\frac{\partial u}{\partial\theta}\frac{\sin\theta}{r}\right)\frac{\cos\theta}{r}$$

$$\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial r^2}\sin^2\theta+\frac{\sin 2\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta}-\frac{\sin 2\theta}{r^2}\frac{\partial u}{\partial\theta}+\frac{\cos^2\theta}{r}\frac{\partial u}{\partial r}+\frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2}$$

得 $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}$

3. $\begin{bmatrix}x_1\\x_2\end{bmatrix}'=\begin{bmatrix}1&-1\\2&-2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}+\frac{e^{-t}}{1+t^2}\begin{bmatrix}1\\2\end{bmatrix}$，$\vec{x}'=A\vec{x}+\vec{b}$

將 A 對角化 $A=SDS^{-1}$

$S=\begin{bmatrix}1&1\\1&2\end{bmatrix}$，$D=\begin{bmatrix}0&0\\0&-1\end{bmatrix}$，$S^{-1}=\begin{bmatrix}2&-1\\-1&1\end{bmatrix}$，令 $\vec{x}=S\vec{y}$ 代入 $\vec{x}'=A\vec{x}+\vec{b}$

$S\vec{y}'=SDS^{-1}\cdot S\vec{y}+\vec{b}$，$\vec{y}'=D\vec{y}+S^{-1}\vec{b}$

$$\begin{bmatrix}y_1\\y_2\end{bmatrix}'=\begin{bmatrix}0&0\\0&-1\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix}+\frac{e^{-t}}{1+t^2}\begin{bmatrix}2&-1\\-1&1\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix},\quad\begin{bmatrix}y_1'\\y_2'\end{bmatrix}=\begin{bmatrix}0\\-y_2\end{bmatrix}+\frac{e^{-t}}{1+t^2}\begin{bmatrix}0\\1\end{bmatrix}$$

$y_1' = 0$， $y_1 = c_1$ ◦ $y_2' = -y_2 + \dfrac{e^{-t}}{1+t^2}$， $(D+1)y_2 = \dfrac{e^{-t}}{1+t^2}$

$y_{2h} = c_2 e^{-t}$， $y_{2p} = \dfrac{1}{D+1}\cdot\dfrac{e^{-t}}{1+t^2} = e^{-t}\dfrac{1}{D}\dfrac{1}{1+t^2} = e^{-t}\cdot\tan^{-1}t$

$y_2 = c_2 e^{-t} + e^{-t}\cdot\tan^{-1}t$ 代入 $\vec{x} = S\vec{y}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 e^{-t} + e^{-t}\cdot\tan^{-1}t \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 2 \end{bmatrix}e^{-t} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}e^{-t}\cdot\tan^{-1}t$$

4. $Y(s) = \dfrac{2s^2 - s}{(s^2+4)^2} = \dfrac{2(s^2+4) - s - 8}{(s^2+4)^2} = \dfrac{2}{s^2+4} - \dfrac{s}{(s^2+4)^2} - \dfrac{8}{(s^2+4)^2}$

$y = L^{-1}[Y(s)] = L^{-1}\left[\dfrac{2}{s^2+4}\right] - L^{-1}\left[\dfrac{s}{(s^2+4)^2}\right] - L^{-1}\left[\dfrac{8}{(s^2+4)^2}\right]$

$L^{-1}\left[\dfrac{s}{(s^2+4)^2}\right] = t\cdot L^{-1}\left[\displaystyle\int_s^\infty \frac{s}{(s^2+4)^2}ds\right] = t\cdot L^{-1}\left[\int_s^\infty \frac{1}{(s^2+4)^2}\frac{1}{2}ds^2\right]$

$= t\cdot L^{-1}\left[-\dfrac{1}{2}\dfrac{1}{s^2+4}\Big|_{s=s}^{s=\infty}\right] = t\cdot L^{-1}\left[\dfrac{1}{2}\dfrac{1}{s^2+4}\right] = \dfrac{1}{4}t\sin 2t$

$L^{-1}\left[\dfrac{8}{(s^2+4)^2}\right] = L^{-1}\left[\dfrac{1}{s}\cdot\dfrac{8s}{(s^2+4)^2}\right] = \displaystyle\int_0^t L^{-1}\left[\frac{8s}{s^2+4)^2}\right]dt = \int_0^t 2t\sin 2t\,dt$

$= \displaystyle\int_0^t td(-\cos 2t) = -t\cos 2t\Big|_0^t + \int_0^t \cos 2t dt = -t\cos 2t + \frac{1}{2}\sin 2t$

得 $y = L^{-1}[Y(s)] = \sin 2t - \dfrac{1}{4}t\sin 2t - \left(-t\cos 2t + \dfrac{1}{2}\sin 2t\right)$

$= \dfrac{1}{2}\sin 2t - \dfrac{1}{4}t\sin 2t + t\cos 2t$

5. $y_m(-p) = y_m(p)$， $y_m'(-p) = y_m'(p)$， $y_n(-p) = y_n(p)$， $y_n'(-p) = y_n'(p)$

$y_m'' + \lambda_m^2 y_m = 0$ (1) ◦ $y_n'' + \lambda_n^2 y_n = 0$ (2) ◦ $\lambda_m \neq \lambda_n$

$(1)\times y_n - (2)\times y_m$， $(y_n y_m'' - y_n'' y_m) + (\lambda_m^2 - \lambda_n^2)y_m y_n = 0$

$(y_m' y_n - y_n' y_m)' = (\lambda_n^2 - \lambda_m^2)y_m y_n$， $\dfrac{d}{dx}(y_m' y_n - y_n' y_m) = (\lambda_n^2 - \lambda_m^2)y_m y_n$

$(\lambda_n^2 - \lambda_m^2)\displaystyle\int_{x=-p}^{p} y_m(x)\cdot y_n(x)dx = \int_{-p}^{p} d(y_m' y_n - y_n' y_m)$

$= [y_m'(p)y_n(p) - y_n'(p)\cdot y_m(p)] - [y_m'(-p)\cdot y_n(-p) - y_n'(-p)\cdot y_m(-p)]$

$= [y_m'(p)\cdot y_n(p) - y_n'(p)\cdot y_m(p)] - [y_m'(p)\cdot y_n(p) - y_n'(p)\cdot y_m(p)] = 0$

$\because\ \lambda_n \neq \lambda_m$, 得 $\int_{-p}^{p} y_m(x)y_n(x)dx = 0$, $\langle y_m(x), y_n(x)\rangle = 0$

$y_m(x)$ 與 $y_n(x)$ 在 $-p < x < p$ 正交, 權重 $W(x) = 1$

6. $f(t)$ is even, 取 Fourier cosine series

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{2},\quad a_0 = \frac{1}{2}\int_0^2 f(t)dt = \frac{1}{2}\int_0^2 (4-t^2)dt = \frac{8}{3}$$

$$a_n = \frac{1}{1}\int_0^2 f(t)\cos\frac{n\pi t}{2}dt = \int_0^2 (4-t^2)\cos\frac{n\pi t}{2}dt = \int_0^2 (4-t^2)d\left(\frac{2}{n\pi}\sin\frac{n\pi t}{2}\right)$$

$$a_n = \frac{2}{n\pi}(4-t^2)\cdot\sin\frac{n\pi t}{2}\bigg|_0^2 + \frac{4}{n\pi}\int_0^2 t\sin\frac{n\pi t}{2}dt$$

$$= \frac{4}{n\pi}\cdot\frac{-2}{n\pi}t\cos\frac{n\pi t}{2}\bigg|_0^2 = \frac{-16}{n^2\pi^2}\cos n\pi = \frac{16}{n^2\pi^2}(-1)^{n+1}$$

得 $f(t) = \frac{8}{3} + \sum_{n=1}^{\infty}\frac{16}{n^2\pi^2}(-1)^{n+1}\cos\frac{n\pi t}{2}$

$$f(t) = \frac{8}{3} + \frac{16}{\pi^2}\left[\frac{1}{1}\cos\frac{\pi t}{2} - \frac{1}{2^2}\cos\frac{2\pi t}{2} + \frac{1}{3^2}\cos\frac{3\pi t}{2} - \frac{1}{4^2}\cos\frac{4\pi t}{2} + \cdots\right]$$

at $t = 0$, 級數收斂到 $f(0) = 4$

$$4 = \frac{8}{3} + \frac{16}{\pi^2}\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\right],\quad \frac{4}{3} = \frac{16}{\pi^2}\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\right]$$

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$$