105年 台灣大學 試題

解題老師：周易

系所：應用力學所(甲、乙組)　科目：工程數學(G)

1. Solve the following ordinary differential equation (14%)

$$y''+\frac{1}{x}y'+\left(1-\frac{1}{4x^2}\right)y=0$$

(1) Rewrite this ODE into a Quasi-Euler Equation (2%)

$(x-x_0)^2\alpha_0 y''+(x-x_0)\beta_0 y'+\gamma(x)y=0$

(2) What is the regular singular point x_0? (2%)

(3) Find the general solution. (10%)

2. Expand the following equation into a Fourier series (10%)

$f(x)=x^2,\ -\pi\le x\le\pi$

(1) Is $f(x)$ an even function or an odd function? (2%)

(2) Find the Fourier series. Check your solution, does it match your answer in question $2(A)$? (8%)

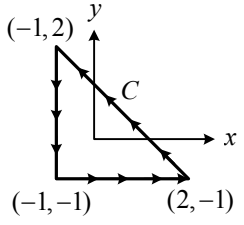
3. Verify the following Laplace transform (9%)

$$L[at+\sin at]=\frac{a^3}{s^2(s^2+a^2)}$$

4. Evaluate the integral along the curve C counter-clockwisely (8%)

$$\oint_C\left[\left(\frac{-y}{x^2+y^2}+5x^2\right)dx+\left(\frac{x}{x^2+y^2}-6y\right)dy\right]$$

where C is the boundary of a triangle with vertices $(-1,-1),(2,-1),(-1,2)$.



5. (1) Find the solution $u(x,t)$ of the following initial-boundary value problem (20%)

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial t} + 4u & ,\ 0 < x < 1,\ t > 0 \\ u(0,t) = 0 & ,\ t > 0 \\ u(1,t) = \sinh 2 & ,\ t > 0 \\ u(x,0) = \sin 3\pi x & ,\ 0 < x < 1 \end{cases}$$

(2) Find $u(1/3,t)$ as $t \to \infty$. (5%)

6. Let A, B and C be $n \times n$ matrices. (34%)

(1) Derive the algebraic decomposition: (4%)

$A = A^{sym} + A^{anti}$

where A^{sym} and A^{anti} are respectively symmetric and antisymmetric matrices.

(2) Denote $\text{trace}(A) = \sum_{i=1}^{n} A_{ii}$. Find $\text{trace}(BC^T)$ where B and C are respectively symmetric and anti-symmetric matrices, and C^T denote the transpose of C.

Note $(BC)_{ik} = \sum_{j=1}^{\infty} B_{ij} C_{jk}$ is a $n \times n$ matrix. Note that BC represents the matrix multiplication of matrices B and C. (5%)

(3) Given B and C being respectively 3×3 symmetric and anti-symmetric matrices, find their associated eigenvalues and eigenvectors (or the governing equations for these eigenvalues and eigenvectors). (10%)

(4) Given B being respectively $n \times n$ symmetric matrix, show that B can be diagonalized. (5%)

(5) Consider the following equation (5%)

$Bx = f$

where B $n \times n$ is a symmetric matrix, and x and f are respectrively are $n \times 1$ vectors. Solve x (B and f are known) utilizing the diagonalizable property of B.

(6) The two matrices B and C are called similar if there exists a nonsingular matrix P such that $B = PCP^{-1}$. Show that B and C possess the same eigenvalues. What is the relationship between the eigenvectors of B and C ?

105年 台灣大學 解答

解題老師：周易

系所：應用力學所(甲、乙組)　科目：工程數學(G)

1. (1) $y''+\frac{1}{x}y'+\left(1-\frac{1}{4x^2}\right)y=0$，$x^2\cdot 4y''+x\cdot 4y'+(4x^2-1)y=0$

為 Quasi Euler ODE

(2) $x_0=0$ 為規則奇異點

(3) $p(x)=\frac{1}{x}$，$Q(x)=1-\frac{1}{4x^2}$，$Q-\frac{1}{4}p^2-\frac{1}{2}p'=\left(1-\frac{1}{4x^2}\right)-\frac{1}{4}\frac{1}{x^2}-\frac{1}{2}\frac{-1}{x^2}=1$

令 $u=\exp\left[\int -\frac{1}{2}pdx\right]=\exp\left[\int -\frac{1}{2}\frac{1}{x}dx\right]=\exp\left(-\frac{1}{2}\ln x\right)=\frac{1}{\sqrt{x}}$

$y=vu=x^{-\frac{1}{2}}v$，$y'=-\frac{1}{2}x^{-\frac{3}{2}}v+x^{-\frac{1}{2}}v'$，$y''=\frac{3}{4}x^{\frac{-5}{2}}v-x^{-\frac{3}{2}}v'+x^{-\frac{1}{2}}v''$

代入 ODE $y''+\frac{1}{x}y'+\left(1-\frac{1}{4x^2}\right)y=0$

$\left(\frac{3}{4}x^{\frac{-5}{2}}v-x^{-\frac{3}{2}}v'+x^{-\frac{1}{2}}v''\right)+\frac{1}{x}\left(-\frac{1}{2}x^{-\frac{3}{2}}v+x^{-\frac{1}{2}}v'\right)+\left(1-\frac{1}{4x^2}\right)x^{-\frac{1}{2}}v=0$

$x^{-\frac{1}{2}}v''+0v'+x^{-\frac{1}{2}}v=0$，$v''+v=0$，$v=c_1\cos x+c_2\sin x$

通解 $y=x^{-\frac{1}{2}}v=c_1\cdot\frac{1}{\sqrt{x}}\cos x+c_2\cdot\frac{1}{\sqrt{x}}\sin x$

2. (1) $f(x)$ is even, 取 Fourier cosine series

(2) $f(x)=a_0+\sum_{n=1}^{\infty}a_n\cos nx$，$a_0=\frac{1}{\pi}\int_0^{\pi}f(x)dx=\frac{1}{\pi}\int_0^{\pi}x^2dx=\frac{1}{3}\pi^2$

$a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos nxdx=\frac{2}{\pi}\int_0^{\pi}x^2\cos nxdx=\frac{2}{n\pi}x^2\sin nx\Big|_0^{\pi}-\frac{4}{n\pi}\int_0^{\pi}x\sin nxdx$

$a_n=\frac{4}{n^2\pi}x\cos nx\Big|_{x=\pi}^{x=0}=\frac{4}{n^2}\cos n\pi=\frac{4}{n^2}(-1)^n$

$f(x)=\frac{1}{3}\pi^2+\sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n\cos nx$

$$f(-x)=\frac{1}{3}\pi^2+\sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n\cos n(-x)=\frac{1}{3}\pi^2+\sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n\cos nx=f(x)$$

$f(x)$ 確實是偶函數

3. $L[at+\sin at]=L[at]+L[\sin at]=\frac{a}{s^2}+\frac{a}{s^2+a^2}=\frac{a[(s^2+a^2)+s^2]}{s^2(s^2+a^2)}=\frac{a^3+(a+1)s^2}{s^2(s^2+a^2)}$

題目有誤

另解：$L^{-1}\left[\frac{a^3}{s^2(s^2+a^2)}\right]=L^{-1}\left[a\left(\frac{1}{s^2}-\frac{1}{s^2+a^2}\right)\right]=at-\sin at$

4. $\oint_C\left(\frac{-y}{x^2+y^2}dx+\frac{x}{x^2+y^2}dy\right)+\oint_C(5x^2dx-6ydy)$

$=\oint_C d\tan^{-1}\frac{y}{x}+\oint_C d\left(\frac{5}{3}x^3-3y^2\right)=\oint_C d\theta+\oint_C d\left(\frac{5}{3}x^3-3y^2\right)$

$=2\pi+0=2\pi$

5. 令 $u(x,t)=w(x,t)+v(x)$，其中 $v(x)=\lim_{t\to\infty}u(x,t)$ 爲穩態解

滿足 $\frac{d^2v}{dx^2}=0+4v$，$v=c_1\cosh 2x+c_2\sinh 2x$

$v(0)=0$，$c_1=0$，$v(1)=\sinh 2$，$c_2=1$，得 $v=\sinh 2x$

將 $u=w+v=w+\sinh 2x$ 代入 PDE $\frac{\partial^2u}{\partial x^2}=\frac{\partial u}{\partial t}+4u$

$$\frac{\partial^2}{\partial x^2}(w+\sinh 2x)=\frac{\partial}{\partial t}(w+\sinh 2x)+4(w+\sinh 2x)$$

$$\frac{\partial^2w}{\partial x^2}=\frac{\partial w}{\partial t}+4w,\quad w(0,t)=0,\quad w(1,t)=0$$

$$w(x,0)=\sin 3\pi x-\sinh 2x$$

取 Fourier sine series

$$w=\frac{2}{1}\sum_{n=1}^{\infty}B_n\sin n\pi x,\quad B_n=\int_0^1 w\sin n\pi x dx$$

$$\int_0^1\frac{\partial^2w}{\partial x^2}\sin n\pi x dx=\int_0^1\frac{\partial w}{\partial t}\sin n\pi x dx+\int_0^1 4w\sin n\pi x dx$$

$$-n^2\pi^2B_n=\frac{d}{dt}B_n+4B_n,\quad [D_t+(n^2\pi^2+4)]B_n=0$$

$B_n=c\exp[-(n^2\pi^2+4)t]$，又 $B_n=\int_0^1 w(x,t)\sin n\pi x dx$

取 $t=0$， $B_n=\int_0^1 w(x,0)\sin n\pi x dx=\int_0^1(\sin 3\pi x-\sinh 2x)\sin n\pi x dx$

$c=\int_0^1 \sin 3\pi x\cdot\sin n\pi x dx-\int_0^1\frac{1}{2}(e^{2x}-e^{-2x})\sin n\pi x dx$

$\because\ \int_0^1 e^{ax}\sin n\pi x dx=\frac{1}{D}e^{ax}\sin n\pi x=e^{ax}\frac{1}{D+a}\sin n\pi x$

$=e^{ax}\frac{D-a}{D^2-a^2}\sin n\pi x=e^{ax}\frac{1}{-n^2\pi^2-a^2}(n\pi\cos n\pi x-a\sin n\pi x)\Big|_{x=0}^{1}$

$=\frac{1}{-n^2\pi^2-a^2}[e^a\cdot n\pi(-1)^n-n\pi]=\frac{n\pi}{n^2\pi^2+a^2}[1-(-1)^n e^a]$

$c=\frac{1}{2}\delta(n-3)-\frac{1}{2}\cdot\frac{n\pi}{n^2\pi^2+4}\{[1-(-1)^n e^2]-[1-(-1)^n e^{-2}]\}$

$c=\frac{1}{2}\delta(n-3)+\frac{n\pi(-1)^n}{n^2\pi^2+4}\sinh 2$

$B_n=\left[\frac{1}{2}\delta(n-3)+\frac{n\pi(-1)^n}{n^2\pi^2+4}\sinh 2\right]\cdot\exp[-(n^2\pi^2+4)t]$

代入 $w=2\sum_{n=1}^{\infty}B_n\sin n\pi x$

$w=2\sum_{n=1}^{\infty}\left[\frac{1}{2}\delta(n-3)+\frac{n\pi(-1)^n}{n^2\pi^2+4}\sinh 2\right]\cdot\exp[-(n^2\pi^2+4)t]\cdot\sin n\pi x$

$w=\exp[-(9\pi^2+4)t]\cdot\sin 3\pi x+2\sum_{n=1}^{\infty}\frac{n\pi(-1)^n}{n^2\pi^2+4}\sinh 2\cdot\exp[-(n^2\pi^2+4)t]\cdot\sin n\pi x$

$u=w(x,t)+v(x)=w(x,t)+\sinh 2x$ 爲解， $w(x,t)$ 如上所示

當 $t\to\infty$， $u\left(\frac{1}{3},t\right)=\sinh\frac{2}{3}=v\left(\frac{1}{3}\right)$

6. (1) $A=\frac{1}{2}(A+A^T)+\frac{1}{2}(A-A^T)$， $A=A^{sym}+A^{anti}$

取 $A^{sym}=\frac{1}{2}(A+A^T)$， $\left[\frac{1}{2}(A+A^T)\right]^T=\frac{1}{2}(A+A^T)$

$A^{anti}=\frac{1}{2}(A-A^T)$， $\left[\frac{1}{2}(A-A^T)\right]^T=\frac{1}{2}(A^T-A)=\frac{-1}{2}(A-A^T)$

(2) $B^T=B$， $b_{ij}=b_{ji}$， $C^T=-C$， $c_{ij}=-c_{ji}$， $c_{ii}=0$， $A_{ij}=\sum_{k=1}^{n}B_{ik}C_{kj}^T$

令 $A=BC^T$，$t_r(BC^T)=t_r(A)=\sum_{i=1}^{n}A_{ii}=\sum_{i=1}^{n}\sum_{k=1}^{n}B_{ik}C_{ik}$

(3)

$B=\begin{bmatrix}a & b & c\\ b & d & e\\ c & e & f\end{bmatrix}$，$B^T=B$，$B\vec{x}=\lambda\vec{x}$，$(B-\lambda I)\vec{x}=\vec{0}$

$$\begin{bmatrix}a-\lambda & b & c\\ b & d-\lambda & e\\ c & e & f-\lambda\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\vec{0}$$

$$|A-\lambda I|=(a-\lambda)(d-\lambda)(f-\lambda)+2bec-[c^2(d-\lambda)+e^2(a-\lambda)+b^2(f-\lambda)]$$
$$=-(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)=0$$

特徵值 $\lambda_1,\lambda_2,\lambda_3$

將 $\lambda_1,\lambda_2,\lambda_3$ 代入 $(A-\lambda I)\vec{x}=\vec{0}$，可得特徵向量 $\vec{x_1},\vec{x_2},\vec{x_3}$

$C=\begin{bmatrix}0 & a & b\\ -a & 0 & d\\ -b & -d & 0\end{bmatrix}$，$C^T=-C$，$C\vec{x}=\lambda\vec{x}$，$(C-\lambda I)\vec{x}=\vec{0}$

$$\begin{bmatrix}-\lambda & a & b\\ -a & -\lambda & d\\ -b & -d & -\lambda\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\vec{0},\quad |c-\lambda I|=-\lambda[\lambda^2+(a^2+b^2+d^2)]$$

特徵值 $\lambda_1=0$，$\lambda_2=-\sqrt{a^2+b^2+d^2}\,i$，$\lambda_3=\sqrt{a^2+b^2+d^2}\,i$

當 $\lambda_1=0$，代入 $(c-\lambda I)\vec{x}=\vec{0}$

$$\begin{bmatrix}0 & a & b\\ -a & 0 & d\\ -b & -d & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\vec{0},\quad \vec{x}=k_1\begin{bmatrix}d\\ -b\\ a\end{bmatrix}$$

當 $\lambda_2=-\sqrt{a^2+b^2+d^2}\,i$，代入 $(c-\lambda I)\vec{x}=\vec{0}$

$$\begin{bmatrix}\sqrt{a^2+b^2+d^2}\,i & a & b\\ -a & \sqrt{a^2+b^2+d^2}\,i & d\\ -b & -d & \sqrt{a^2+b^2+d^2}\,i\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\vec{0},$$

$$\vec{x}=k_2\begin{bmatrix}db+ia\sqrt{a^2+b^2+d^2}\\ a^2+d^2\\ ab-di\sqrt{a^2+b^2+d^2}\end{bmatrix}$$

當 $\lambda_3 = \sqrt{a^2+b^2+d^2}\,i$

$$\vec{x} = k_3 \begin{bmatrix} db - ia\sqrt{a^2+b^2+d^2} \\ a^2+d^2 \\ ab + id\sqrt{a^2+b^2+d^2} \end{bmatrix}$$

(4) 利用 Schur's 定理, 所有矩陣都可公正上三角化 $B = SUS^*$, $S^* = S^{-1}$

U 爲上三角矩陣, $B^T = (SUS^*)^T = (S^*)^T U^T S^T = \overline{S}U^T S^T$

$\because B^T = B$, 得 $S = \overline{S}$, S 爲實數矩陣, $S^* = (\overline{S})^T = S^T = S^{-1}$, S 爲正交矩陣

$U = U^T$, $\because U$ 是上三角矩陣, 得 $U = D$ 爲對角矩陣

$B = SDS^T$, $S^T = S^{-1}$ 故實對稱矩陣 B 可正交對角化

(5) $B\vec{x} = \vec{f}$, $B = SDS^{-1}$, $S^{-1} = S^T$, $SDS^{-1}\cdot\vec{x} = \vec{f}$, $DS^{-1}\vec{x} = S^{-1}\vec{f}$

$DS^T\vec{x} = S^T\vec{f}$

設 B 之特徵值不爲零, B 可逆, D 可逆, $\vec{x} = SD^{-1}S^T\vec{f}$

若 B 不可逆, B 之特徵值有零, D 不可逆, $\vec{x}$ 爲無解 or 有參數解,

令 $\vec{y} = S^{-1}\vec{x}$, $\vec{g} = S^{-1}\vec{f}$, $D\vec{y} = \vec{g}$

當 $\lambda_i = 0$, $i = 1 \sim n$

若 $g_i \neq 0$, 則 y_i 無解, $\vec{x} = S\vec{y}$ 無解

若 $g_i = 0$, 則 y_i 爲任意常數 c_i, $\vec{x} = S\vec{y}$ 爲解, 有參數解

(6) $|B-\lambda I| = |PCP^{-1} - \lambda I| = |PCP^{-1} - \lambda PIP^{-1}| = |P(C-\lambda I)\cdot P^{-1}|$

$= |P|\cdot|C-\lambda I|\cdot|P^{-1}| = |C-\lambda I|$

B 與 C 有相同特徵多項式, 故 B 與 C 有相同特徵值

令 B 與 C 之特徵向量分別爲 $\vec{x}$ 與 $\vec{y}$, $B\vec{x} = \lambda\vec{x}$, $C\vec{y} = \lambda\vec{y}$

$PCP^{-1}\vec{x} = \lambda\vec{x}$, $CP^{-1}\vec{x} = P^{-1}\lambda\vec{x}$, $C(P^{-1}\vec{x}) = \lambda(P^{-1}\vec{x})$, 得 $\vec{y} = P^{-1}\vec{x}$