105年 台灣大學 試題

解題老師：周易

系所：環境工程所(科學與工程甲、乙組)　科目：工程數學(H)

1. Find the general solutions of the differential equations:

(1) $x^2y''-2xy'+2y=0$ (10%)

(2) $y''-4y'+5y=e^{2x}\csc x$ (10%)

2. Find the solution of $x''+2x'+2x=g(t)$, $x(0)=0$, $x'(0)=0$

Where, $g(t)=1$ for $0\le t\le 1$, $g(t)=0$ for $t>1$, by Laplace transform. (15%)

3. Write down the "Divergence theorem" and "Stokes's theorem" in vector analysis, and with adequate explanation (better with the aid of figures). (15%)

4. Consider a $1.0\times10^7\,\mathrm{m}^3$ lake fed by a polluted stream having a flow rate of 5.0 m^3/s and pollution concentration equal to 10.0 mg/L. There is also a sewage outfall that discharges 0.5 m^3/s of wastewater having a pollutant concentration of 100 gm/L. The stream and sewage wastes have a reaction rate coefficient of 0.20/day. Assuming the pollution is completely mixed in the lake, and assuming no evaporation or other water losses or gains, (1) find the steady-state pollutant concentration. (10%) If we divert the sewage outfall around the lake, eliminating it as a source of pollution, assuming complete-mix conditions, (2) find the concentration of pollution in the lake one week after the diversion. (15%)

5. One way to disaggregate environmental impacts of human activity is with the following product:

$$\text{Impacts}=(\text{Population})\times(\text{Affluence})\times(\text{Technology})$$

Where, affluence is indicated by per capita energy demand, while technology is represented by the carbon emission per unit of energy. Each factor is itself growing exponentially; that is, the rate of change of the quantity N is proportional to N $(dN/dt=rN)$. The proportionality constant r is called the rate of growth and has unit of (time^{-1}). If per capita energy demand increases at 1.5% per year, fossil fuel emissions of carbon per unit of energy decreases at 0.5% per year, and population grows at 1.0% per

year, (1) how long would it take before we are emitting carbon at twice the current rate? (15%) (2) At that point, by what fraction would per capita energy demand have increased? (10%)

105年 台灣大學 解答

解題老師：周易

系所：環境工程所(科學與工程甲、乙組)　科目：工程數學(H)

1. (1) 令 $x=e^t$，$t=\ln x$ 代入 $x^2y''-2xy'+2y=0$，$D_t(D_t-1)y-2D_ty+2y=0$

$(D_t^2-3D_t+2)y=0$，$(D_t-1)(D_t-2)y=0$，$y=c_1e^t+c_2e^{2t}=c_1x+c_2x^2$

(2) $(D^2-4D+5)y=e^{2x}\csc x$，當 $D^2-4D+5=0$，$(D-2)^2=-1$，$D=2\pm i$

$y_h=c_1e^{2x}\cos x+c_2e^{2x}\sin x$，

$$y_p=\frac{1}{D^2-4D+5}e^{2x}\csc x=e^{2x}\frac{1}{(D+2)^2-4(D+2)+5}\csc x$$

$$y_p=e^{2x}\frac{1}{D^2+1}\csc x=e^{2x}\frac{1}{(D-i)(D+i)}\csc x=e^{2x}\frac{1}{2i}\left(\frac{1}{D-i}-\frac{1}{D+i}\right)\csc x$$

$$y_p=\frac{1}{2i}e^{2x}[e^{ix}\int e^{-ix}\csc x dx-e^{-ix}\int e^{ix}\csc x dx]$$

$$y_p=\frac{1}{2i}e^{2x}\left[e^{ix}\int(\cos x-i\sin x)\frac{1}{\sin x}dx-e^{-ix}\int(\cos x+i\sin x)\frac{1}{\sin x}dx\right]$$

$$=\frac{1}{2i}e^{2x}[e^{ix}(\ln\sin x-ix)-e^{-ix}(\ln\sin x+ix)]$$

$$=e^{2x}\cdot\ln\sin x\cdot\frac{1}{2i}(e^{ix}-e^{-ix})-xe^{2x}\frac{1}{2}(e^{ix}+e^{-ix})$$

$$=e^{2x}\cdot\ln\sin x\cdot(\sin x)-xe^{2x}\cos x$$

通解　$y=c_1e^{2x}\cos x+c_2e^{2x}\sin x+\sin x\cdot e^{2x}\ln(\sin x)-xe^{2x}\cos x$

2. $g(t)=[u(t)-u(t-1)]\cdot 1$，$L[g(t)]=\frac{1}{s}e^{-0s}-\frac{1}{s}e^{-s}$

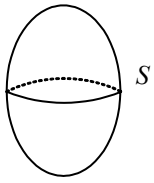
$L[x''+2x'+2x]=L[g(t)]$，$s^2X+2sX+2X=\frac{1}{s}(e^{-0s}-e^{-s})$

$$X=\frac{1}{s(s^2+2s+2)}(e^{-0s}-e^{-s})=\left(\frac{1}{2}\frac{1}{s}+\frac{\frac{-1}{2}s-1}{s^2+2s+2}\right)(e^{-0s}-e^{-s})$$

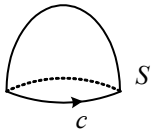
$$X=\frac{1}{2}\cdot\frac{1}{s}(e^{-0s}-e^{-s})-\frac{1}{2}\frac{(s+1)+1}{(s+1)^2+1}(e^{-0s}-e^{-s})$$

$$x(t)=L^{-1}[X(s)]=\frac{1}{2}[u(t)-u(t-1)]-\frac{1}{2}u(t)\cdot e^{-t}(\cos t+\sin t)$$
$$+\frac{1}{2}u(t-1)\cdot e^{-(t-1)}\cdot[\cos(t-1)+\sin(t-1)]$$

3. (1) S 爲空間中之封閉曲面, R 爲 S 所包圍區域
$\vec{F}$ 在 S 上和 S 內沒有奇異點, 則
$\iint_S \vec{F}\cdot\vec{n}dA=\iiint_R \nabla\cdot\vec{F}dV$ 這是散度定理



(2) S 爲空間中有開口曲面, C 爲曲面 S 之邊界曲線
$\vec{F}$ 在 S 上沒有奇異點, 則
$\oint_C \vec{F}\cdot d\vec{r}=\iint_S(\nabla\times\vec{F})\cdot\vec{n}dA$ 這是 stoke's 定理



第 4, 5 題爲環工專業科目，請參考相關書籍