105 年 台灣大學 試題

解題老師：周易

系所：工科海洋系(丁組)　科目：線性代數(B)

1. Factor A into LU, and find the solution for: (20%)

$$A\vec{x}=\begin{bmatrix}2&-1&&\\-1&2&-1&\\&-1&2&-1\\&&-1&2\end{bmatrix}\begin{bmatrix}x\\y\\z\\t\end{bmatrix}=\begin{bmatrix}0\\0\\3\\-1\end{bmatrix}$$

2. Which of the following are subspaces of R^{∞} with reason? (20%)

(A) All sequences like $(1,0,1,0,\ldots)$ that include infinitely many zeros.

(B) All sequences $(x_1, x_2, \ldots)$ with $x_j = 0$ from some point onward.

(C) All decreasing sequences: $x_{j+1} \le x_j$ for each j.

(D) All convergent sequences: the x_j have a limit as $j \to \infty$.

(E) All geometric progressions $(x_1, kx_1, k^2x_1, \ldots)$ allowing all k and x_1.

3. Find a best approximation to $y = x^5$ by a straight line between $x = -1$ and $x = 1$. (20%)

4. Find orthogonal vectors A, B, C by Gram-Schmidt process from $a = (1,-1,0,0)$, $b = (0,1,-1,0)$, $c = (0,0,1,-1)$. (10%)

5 Find the eigenvalues and eigenvectors of

$$A=\begin{bmatrix}0&-i&0\\i&1&i\\0&-i&0\end{bmatrix}.\ (10\%)$$

6. $A=\begin{bmatrix}5&4\\4&5\end{bmatrix}$ and show A is positive definite. If $\vec{x}^{T}A\vec{x}=1$, draw the tilted ellipse and find the half-lengths of its axes. (20%)

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1.

$$a=\begin{bmatrix}2&-1&0&0\\-1&2&-1&0\\0&-1&2&-1\\0&0&-1&2\end{bmatrix}\xrightarrow{R_{12}\left(\frac{1}{2}\right)}\begin{bmatrix}2&-1&0&0\\0&\frac{3}{2}&-1&0\\0&-1&2&-1\\0&0&-1&2\end{bmatrix}$$

$$\xrightarrow{R_{23}\left(\frac{2}{3}\right)}\begin{bmatrix}2&-1&0&0\\0&\frac{3}{2}&-1&0\\0&0&\frac{4}{3}&-1\\0&0&-1&2\end{bmatrix}\xrightarrow{R_{34}\left(\frac{3}{4}\right)}\begin{bmatrix}2&-1&0&0\\0&\frac{3}{2}&-1&0\\0&0&\frac{4}{3}&-1\\0&0&0&\frac{5}{4}\end{bmatrix}=U$$

$$A\xleftarrow{R_{12}^{-1}\left(\frac{1}{2}\right),\,R_{23}^{-1}\left(\frac{2}{3}\right),\,R_{34}^{-1}\left(\frac{3}{4}\right)}U$$

$$A\xleftarrow{R_{12}\left(\frac{-1}{2}\right),\,R_{23}\left(\frac{-2}{3}\right),\,R_{34}\left(\frac{-3}{4}\right)}U$$

$$A=ER_{12}\left(\frac{-1}{2}\right)\cdot ER_{23}\left(\frac{-2}{3}\right)\cdot ER_{34}\left(\frac{-3}{4}\right)U=LU$$

$$L=ER_{12}\left(\frac{-1}{2}\right)\cdot ER_{23}\left(\frac{-2}{3}\right)\cdot ER_{34}\left(\frac{-3}{4}\right)=ER_{12}\left(\frac{-1}{2}\right)\cdot ER_{23}\left(\frac{-2}{3}\right)\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&\frac{-3}{4}&1\end{bmatrix}$$

$$L=\begin{bmatrix}1&0&0&0\\ \frac{-1}{2}&1&0&0\\ 0&\frac{-2}{3}&1&0\\ 0&0&\frac{-3}{4}&1\end{bmatrix},\quad U=\begin{bmatrix}2&-1&0&0\\ 0&\frac{3}{2}&-1&0\\ 0&0&\frac{4}{3}&-1\\ 0&0&0&\frac{5}{4}\end{bmatrix},\quad A=LU$$

$A\vec{x}=\vec{b}$, $\vec{b}=\begin{bmatrix}0\\0\\3\\-1\end{bmatrix}$, $LU\vec{x}=\vec{b}$, 令 $\vec{y}=U\vec{x}$, 得 $L\vec{y}=\vec{b}$

$$\begin{bmatrix}1&0&0&0\\ \frac{-1}{2}&1&0&0\\ 0&\frac{-2}{3}&1&0\\ 0&0&\frac{-3}{4}&1\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\\y_4\end{bmatrix}=\begin{bmatrix}0\\0\\3\\-1\end{bmatrix}\qquad \begin{aligned}&y_1=0\\ &\tfrac{-1}{2}y_1+y_2=0 \quad ,\ y_2=0\\ &\tfrac{-2}{3}y_2+y_3=3 \quad ,\ y_3=3\\ &-\tfrac{3}{4}y_3+y_4=-1 \quad ,\ y_4=\tfrac{5}{4}\end{aligned}$$

代入 $U\vec{x}=\vec{y}$

$$\begin{bmatrix}2&-1&0&0\\ 0&\frac{3}{2}&-1&0\\ 0&0&\frac{4}{3}&-1\\ 0&0&0&\frac{5}{4}\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}0\\0\\3\\\frac{5}{4}\end{bmatrix},\qquad \begin{aligned}&2x_1-x_2=0\\ &\tfrac{3}{2}x_2-x_3=0\\ &\tfrac{4}{3}x_3-x_4=3\\ &\tfrac{5}{4}x_4=\tfrac{5}{4}\end{aligned}$$

$x_4=1$, $x_3=3$, $x_2=2$, $x_1=1$

2. (A) $U=\{(a,0,a,0,\ldots)\}$, 取 $\vec{f}=(a,0,a,0,\ldots)=a(1,0,1,0,\ldots)$, 得 $(1,0,1,0,\ldots)$ 爲 U 基底, U 爲 R^{∞} 中一維子空間

(B) $U=\{(x_1,x_2,\ldots,x_{j-1},0,0,\ldots,0)\}$, $\forall\vec{x}=(x_1,x_2,\ldots,x_{j-1},0,\ldots,0)\in U$

$\vec{x}=x_1(1,0,0,\ldots)+x_2(0,1,0,\ldots)+\cdots+x_{j-1}(0,0,\ldots,0,1,0,\ldots)\in U$

$\{(1,0,0,\ldots),(0,1,0,\ldots),\ldots,(0,0,\ldots,0,1,0,\ldots)\}$ 爲 U 基底,

U 存在基底, U 爲向量空間, $\dim U=j-1$

U 爲 R^{∞} 中 $(j-1)$ 維子空間

(C) $U=\{(x_1,x_2,\ldots,x_j,x_{j+1},\ldots x_n,\ldots)\}$

取 $\vec{x}=(2,1,0,0,\ldots,0,\ldots)\in U$, $-2\vec{x}=(-4,-2,0,0,\ldots)\notin U$

U 中之向量沒有滿足數乘封閉性, 加法封閉性成立, U 不是向量空間,

U 不是 R^{∞} 子空間

(D) $U=\{(x_1,x_2,\ldots,x_j,\ldots)\}$, $x_j=k$ as $j\to\infty$, k is finite

取 $\vec{x}=(x_1,x_2,\ldots,x_j,\ldots)\in U$, $\vec{y}=(y_1,y_2,\ldots,y_j,\ldots)\in U$

$\lim_{j\to\infty} x_j=k$, $\lim_{j\to\infty} y_j=\alpha$

$\vec{x}+\vec{y}=(x_1+y_1,x_2+y_2,\ldots,x_j+y_j,\ldots)$

$c\vec{x}=(cx_1,cx_2,\ldots,cx_j,\ldots)$, c is constant

$\because$ $\lim_{j\to\infty} x_j+y_j=k+\alpha$ is finite, $\lim_{j\to\infty} cx_j$ is finite

$\vec{x}+\vec{y}\in U$, $c\vec{x}\in U$, U 中向量滿足加法與數乘封閉性

U 爲 R^{∞} 中子空間

(E) $U=\{x_1(1,k,k^2,k^3,\ldots),k,x_1\in R\}$

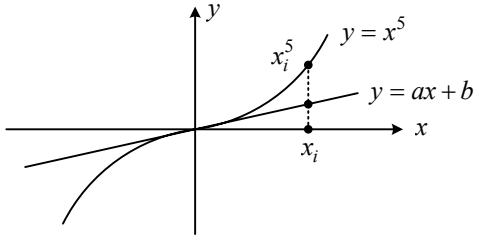
取 $\vec{x}=(1,2,2^2,2^3,\ldots)\in U$, $\vec{y}=(1,3,3^2,3^3,\ldots)\in U$

$\vec{x}+\vec{y}=(2,5,13,35,\ldots)\notin U$,

U 中向量沒有滿足加法封閉性, U 不是向量空間,

U 不是 R^{∞} 子空間

3.



曲線 $y=x^5$ 上任意點 (x_i,y_i) , $y_i=x_i^5$ 到直線 $y=ax+b$ 之 y 方向距離平方和最小,

則 $y=ax+b$ 爲均方近似直線

令 $s=\sum_{-1<x_i<1}(ax_i+b-x_i^5)^2$, 無限多點之級數和, 成爲積分

$s=\int_{-1}^{1}(ax+b-x^5)^2\,dx$, $s=s(a,b)$

當 $\frac{\partial s}{\partial a}=\int_{-1}^{1}\frac{\partial}{\partial a}(ax+b-x^5)^2\,dx=0$

$\frac{\partial s}{\partial b}=\int_{-1}^{1}\frac{\partial}{\partial b}(ax+b-x^5)^2dx=0$，$s$ 爲最小

$\int_{-1}^{1}2(ax+b-x^5)\cdot x\,dx=0$，$\int_{-1}^{1}2(ax+b-x^5)dx=0$

$2\cdot 2\left(\frac{1}{3}a+0-\frac{1}{7}\right)=0$，$a=\frac{3}{7}$ ◦ $b=0$，得 $y=\frac{3}{7}x$ 爲均方近似直線

4. 令 $\vec{\phi_1}=\vec{a}=(1,-1,0,0)$，$\|\vec{\phi_1}\|^2=2$ ◦ $\vec{\phi_2}=\vec{b}-\frac{\langle\vec{b},\vec{\phi_1}\rangle}{\langle\vec{\phi_1},\vec{\phi_1}\rangle}\vec{\phi_1}$

$\vec{\phi_2}=(0,1,-1,0)-\frac{-1}{2}(1,-1,0,0)=\frac{1}{2}(1,1,-2,0)$，$\|\vec{\phi_2}\|^2=\frac{3}{2}$

$\vec{\phi_3}=\vec{c}-\frac{\langle\vec{c},\vec{\phi_1}\rangle}{\langle\vec{\phi_1},\vec{\phi_1}\rangle}\vec{\phi_1}-\frac{\langle\vec{c},\vec{\phi_2}\rangle}{\langle\vec{\phi_2},\vec{\phi_2}\rangle}\vec{\phi_2}$

$\vec{\phi_3}=(0,0,1,-1)-0\vec{\phi_1}-\frac{-1}{\frac{3}{2}}\frac{1}{2}(1,1,-2,0)=\frac{1}{3}(1,1,1,-3)$

令 $\vec{A}=\vec{\phi_1}=(1,-1,0,0)$，$\vec{B}=\vec{\phi_2}=\frac{1}{2}(1,1,-2,0)$，$\vec{c}=\vec{\phi_3}=\frac{1}{3}(1,1,1,-3)$

則 $\{\vec{A},\vec{B},\vec{C}\}$ 爲正交向量

5. $A\vec{x}=\lambda\vec{x}$，$(A-\lambda I)\vec{x}=\vec{0}$

$$\begin{bmatrix}-\lambda & -i & 0\\ i & 1-\lambda & i\\ 0 & -i & -\lambda\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\vec{0},\quad |A-\lambda I|=-\lambda(\lambda-2)(\lambda+1)$$

特徵值 $\lambda_1=-1$，$\lambda_2=0$，$\lambda_3=2$

當 $\lambda_1=-1$　　eigenvectors

$$\begin{bmatrix}1 & -i & 0\\ i & 2 & i\\ 0 & -i & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\vec{0},\qquad \vec{x}=c_1\begin{bmatrix}i\\1\\i\end{bmatrix}=c_1\overrightarrow{X_1}$$

當 $\lambda_2=0$

$$\vec{x}=c_2\begin{bmatrix}1\\0\\-1\end{bmatrix}=c_2\overrightarrow{X_2},$$

當 $\lambda_3 = 2$

$$\begin{bmatrix} -2 & -i & 0 \\ i & -1 & i \\ 0 & -i & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{0}, \qquad \vec{x} = c_3\begin{bmatrix} i \\ -2 \\ i \end{bmatrix} = c_3\overrightarrow{X_3}$$

說明：

$$\left\langle \overrightarrow{X_1}, \overrightarrow{X_2} \right\rangle = \overrightarrow{X_2}^{*}\overrightarrow{X_1} = 0,\ \left\langle \overrightarrow{X_1}, \overrightarrow{X_3} \right\rangle = \overrightarrow{X_3}^{*}\overrightarrow{X_1} = 0,\ \left\langle \overrightarrow{X_2}, \overrightarrow{X_3} \right\rangle = \overrightarrow{X_3}^{*}\overrightarrow{X_2} = 0$$

$\overrightarrow{X_1}$, $\overrightarrow{X_2}$, $\overrightarrow{X_3}$ 爲正交特徵向量

6. $A\vec{u} = \lambda\vec{u}$, $(A - \lambda I)\vec{u} = \vec{0}$

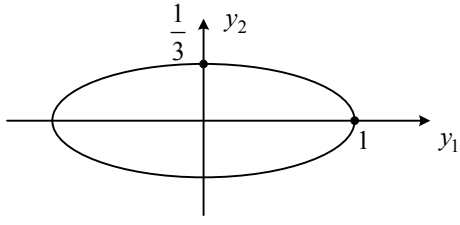
$$\begin{bmatrix} 5-\lambda & 4 \\ 4 & 5-\lambda \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \vec{0},\ \lambda_1 = 1,\ \vec{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix},\ \lambda_2 = 9,\ \vec{u} = c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$A^T = A$，且 A 之特徵值都正，A 爲正定矩陣

取 $S = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$，$S^T = S^{-1}$，$D = \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}$，$A = SDS^{-1}$爲正交對角化

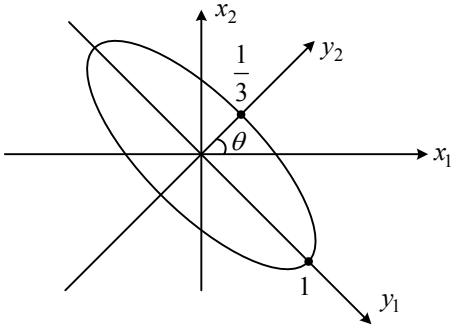
令 $\vec{x} = S\vec{y}$ 代入 $\vec{x}^T A\vec{x} = 1$，$(S\vec{y})^T SDS^{-1} \cdot S\vec{y} = 1$，$\vec{y}^T S^T SD\vec{y} = 1$

$\vec{y}^T D\vec{y} = 1$，$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}^T\begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 1$，${y_1}^2 + 9{y_2}^2 = 1$，半軸長 $\frac{1}{3}, 1$



$\vec{y} = S^{-1}\vec{x} = S^T\vec{x}$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix},\ \begin{cases} y_1 = \frac{1}{\sqrt{2}}x_1 - \frac{1}{\sqrt{2}}x_2 \\ y_2 = \frac{1}{\sqrt{2}}x_1 + \frac{1}{\sqrt{2}}x_2 \end{cases},\ \begin{cases} \overrightarrow{e_1} = \frac{1}{\sqrt{2}}\vec{i} - \frac{1}{\sqrt{2}}\vec{j} \\ \overrightarrow{e_2} = \frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j} \end{cases}$$



$\theta = 45^\circ$