105年 台灣大學 試題

解題老師：周易

系所：生機系(丙組)　科目：線性代數(C)

1. 請計算由直角座標 (x, y) 轉換爲極座標 polar coordinates (r, θ) 之 Jacobian 值，並說明其幾何意義。其相關式： $x = r\cos\theta$， $y = r\sin\theta$ (15%)

2. 以高氏消去法　求算 x_1, x_2, x_3 之值 (15%)

$$\begin{bmatrix} 2 & 6 & 1 \\ 1 & 2 & -1 \\ 5 & 7 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \\ 9 \end{bmatrix}$$

3. 解下列方程式組：(20%)

$$\frac{dx}{dt} = -4x + 2y, \quad \frac{dy}{dt} = -\frac{5}{2}x + 2y$$

4. 試求算下列的行列式(n 階)之值：(15%)

$$\Delta_n = \begin{vmatrix} x & y & y & \cdots & y & y \\ z & x & y & \cdots & y & y \\ z & z & x & \cdots & y & y \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ z & z & z & \cdots & x & y \\ z & z & z & \cdots & z & x \end{vmatrix}_n = ? \quad \text{其中 } y \neq z$$

5. 試求算下列的 $\boldsymbol{A}$ 矩陣之秩(rank)：(15%)

$$\boldsymbol{A} = \begin{bmatrix} 1 & -2 & 3 & -1 & -1 & -2 \\ 2 & -1 & 1 & 0 & -2 & -2 \\ -2 & -5 & -8 & -4 & 3 & -1 \\ 6 & 0 & -1 & 2 & -7 & -5 \\ -1 & -1 & 1 & -1 & 2 & 1 \end{bmatrix}$$

6. 令 F 爲複數的子體(subfield)， T 爲從 F^3 至 F^3 的函數。定義 T 爲

$$T(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3, 2x_1 + x_2, -x_1 - 2x_2 + 2x_3)$$

(1) 試證： T 爲線性變換(linear transformation)。(6%)

(2) 若 (a, b, c) 爲 F^3 的一向量，試問 a, b, c 在何種條件下， (a, b, c) 屬於 T 的值域

(range)？(5%)又，T 的秩(rank)爲何？(2%)

(3) 試問 a, b, c 在何種條件下，(a, b, c) 屬於 T 的零空間(null space)？(5%)
又，T 的零維數(即， Nullity(T))爲何？(2%)

105年 台灣大學 解答

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1.

$$J=\frac{\partial(x,\,y)}{\partial(r,\,\theta)}=\begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{vmatrix}=\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}=r\cos^2\theta+r\sin^2\theta=r$$

J 爲 (x,y) 座標轉到 (r,θ) 座標之座標轉換因子

2.

$$\left[\begin{array}{ccc|c}2&6&1&7\\1&2&-1&-1\\5&7&-4&9\end{array}\right]\xrightarrow{R_{12}}\left[\begin{array}{ccc|c}1&2&-1&-1\\2&6&1&7\\5&7&-4&9\end{array}\right]\xrightarrow[R_{13}(-5)]{R_{12}(-2)}\left[\begin{array}{ccc|c}1&2&-1&-1\\0&2&3&9\\0&-3&1&14\end{array}\right]$$

$$\xrightarrow{R_2\left(\frac{1}{2}\right)}\left[\begin{array}{ccc|c}1&2&-1&-1\\0&1&\frac{3}{2}&\frac{9}{2}\\0&-3&1&14\end{array}\right]\xrightarrow[R_{23}(3)]{R_{21}(-2)}\left[\begin{array}{ccc|c}1&0&-4&-10\\0&1&\frac{3}{2}&\frac{9}{2}\\0&0&\frac{11}{2}&\frac{55}{2}\end{array}\right]$$

$$\xrightarrow{R_3\left(\frac{2}{11}\right)}\left[\begin{array}{ccc|c}1&0&-4&-10\\0&1&\frac{3}{2}&\frac{9}{2}\\0&0&1&5\end{array}\right]\xrightarrow[R_{32}\left(\frac{-3}{2}\right)]{R_{31}(4)}\left[\begin{array}{ccc|c}1&0&0&10\\0&1&0&-3\\0&0&1&5\end{array}\right],\quad\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}10\\-3\\5\end{bmatrix}$$

3.

$$\begin{bmatrix}x\\y\end{bmatrix}'=\begin{bmatrix}-4 & 2\\ \frac{-5}{2} & 2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix},\quad \vec{r}\,'=A\vec{r},\ \text{將 } A \text{ 對角化}$$

$$A\vec{u}=\lambda\vec{u},\quad (A-\lambda I)\vec{u}=\vec{0}$$

$$\begin{bmatrix}-4-\lambda & 2\\ \frac{-5}{2} & 2-\lambda\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix}=\vec{0},\quad |A-\lambda I|=\lambda^2+2\lambda-3$$

$\lambda_1=-3$, $\vec{u}=c_1\begin{bmatrix}2\\1\end{bmatrix}$, $\lambda_2=1$, $\vec{u}=c_2\begin{bmatrix}2\\5\end{bmatrix}$

$S=\begin{bmatrix}2&2\\1&5\end{bmatrix}$, $D=\begin{bmatrix}-3&0\\0&1\end{bmatrix}$, $A=SDS^{-1}$, 令 $\vec{r}=S\vec{r_1}=S\begin{bmatrix}x_1\\y_1\end{bmatrix}$

代入 $\vec{r}'=A\vec{r}$, $S\vec{r_1}'=SDS^{-1}\cdot S\vec{r_1}$, $\vec{r_1}'=D\vec{r_1}$,

$\begin{bmatrix}x_1\\y_1\end{bmatrix}'=\begin{bmatrix}-3&0\\0&1\end{bmatrix}\begin{bmatrix}x_1\\y_1\end{bmatrix}$, $\begin{cases}x_1'=-3x_1 & , \ x_1=c_1e^{-3t}\\ y_1'=y_1 & , \ y_1=c_2e^{t}\end{cases}$

$\vec{r}=S\vec{r_1}=\begin{bmatrix}2&2\\1&5\end{bmatrix}\begin{bmatrix}c_1e^{-3t}\\c_2e^{t}\end{bmatrix}$

$\begin{bmatrix}x\\y\end{bmatrix}=c_1\begin{bmatrix}2\\1\end{bmatrix}e^{-3t}+c_2\begin{bmatrix}2\\5\end{bmatrix}e^{t}$

4.

$$\Delta_n \xrightarrow[\substack{R_2\left(\frac{1}{x}\right)\\ \vdots \\ R_n\left(\frac{1}{x}\right)}]{R_1\left(\frac{1}{x}\right)} x^n\cdot\begin{vmatrix}1&\alpha&\alpha&\cdots&\alpha&\alpha\\ \beta&1&\alpha&\cdots&\alpha&\alpha\\ \beta&\beta&1&\cdots&\alpha&\alpha\\ \vdots&\vdots&\vdots&&\vdots&\vdots\\ \beta&\beta&\beta&\cdots&1&\alpha\\ \beta&\beta&\beta&\cdots&\beta&1\end{vmatrix}_{n\times n}, \quad \begin{matrix}\alpha=\frac{y}{x}\\ \beta=\frac{z}{x}\end{matrix}$$

$$\xrightarrow[\substack{R_{13}(-\beta)\\ \vdots \\ R_{1n}(-\beta)}]{R_{12}(-\beta)} x^n\begin{vmatrix}1&\alpha&\alpha&\cdots&\alpha&\alpha\\ 0&1-\alpha\beta&\alpha-\alpha\beta&\cdots&\alpha-\alpha\beta&\alpha-\alpha\beta\\ 0&\beta-\alpha\beta&1-\alpha\beta&\cdots&\alpha-\beta\alpha&\alpha-\alpha\beta\\ \vdots&\vdots&\vdots&&\vdots&\vdots\\ 0&\beta-\alpha\beta&\beta-\alpha\beta&\cdots&1-\alpha\beta&\alpha-\alpha\beta\\ 0&\beta-\alpha\beta&\beta-\alpha\beta&\cdots&\beta-\alpha\beta&1-\alpha\beta\end{vmatrix}=x^n\cdot 1\cdot A_{11}$$

$$=x^n\begin{vmatrix}1-\alpha\beta&\alpha-\alpha\beta&\cdots\cdots&\alpha-\alpha\beta&\alpha-\alpha\beta\\ \beta-\alpha\beta&1-\alpha\beta&&\alpha-\alpha\beta&\alpha-\alpha\beta\\ \vdots&\vdots&&\vdots&\vdots\\ \beta-\alpha\beta&\beta-\alpha\beta&&1-\alpha\beta&\alpha-\alpha\beta\\ \beta-\alpha\beta&\beta-\alpha\beta&&\beta-\alpha\beta&1-\alpha\beta\end{vmatrix}$$

和原始 Δ_n 形狀相同

$$\xrightarrow[\vdots]{\begin{matrix}R_1\left(\frac{1}{1-\alpha\beta}\right)\\ \\ R_n\left(\frac{1}{1-\alpha\beta}\right)\end{matrix}} x^n(1-\alpha\beta)^{n-1}\begin{vmatrix}1 & \alpha_1 & \cdots\cdots & \alpha_1 & \alpha_1\\ \beta_1 & 1 & \cdots\cdots & \alpha_1 & \alpha_1\\ \vdots & \vdots & & \vdots & \vdots\\ \beta_1 & \beta_1 & \cdots\cdots & 1 & \alpha_1\\ \beta_1 & \beta_1 & \cdots\cdots & \beta_1 & \beta_1\end{vmatrix}_{n-1}, \quad \alpha_1=\frac{\alpha(1-\beta)}{1-\alpha\beta}, \quad \beta_1=\frac{\beta(1-\alpha)}{1-\alpha\beta}$$

同(1)式相同

$$\Delta_n = x^n(1-\alpha\beta)^{n-1}(1-\alpha_1\beta_1)^{n-2}\begin{vmatrix}1 & \alpha_2 & \cdots\cdots & \alpha_2 & \alpha_2\\ \beta_2 & 1 & \cdots\cdots & \alpha_2 & \alpha_2\\ \vdots & \vdots & & \vdots & \vdots\\ \beta_2 & \beta_2 & \cdots\cdots & 1 & \alpha_2\\ \beta_2 & \beta_2 & \cdots\cdots & \beta_2 & \beta_2\end{vmatrix}_{n-2}$$

$$\alpha_2=\frac{\alpha_1(1-\beta_1)}{1-\alpha_1\beta_1},\quad \beta_2=\frac{\beta_1(1-\alpha_1)}{1-\alpha_1\beta_1}$$

$$\Delta_n = x^n(1-\alpha\beta)^{n-1}(1-\alpha_1\beta_1)^{n-2}(1-\alpha_2\beta_2)^{n-3}\cdots(1-\alpha_{n-1}\beta_{n-1})$$

$$\beta_{i+1}=\frac{\beta_i(1-\alpha_i)}{1-\alpha_i\beta_i},\quad \alpha_{i+1}=\frac{\alpha_i(1-\beta_i)}{1-\alpha_i\beta_i},\quad i=1,2,3,\ldots$$

$$\alpha=\frac{y}{x},\quad \beta=\frac{z}{x}$$

5.

$$A=\begin{bmatrix}1 & -2 & 3 & -1 & -1 & -2\\ 2 & -1 & 1 & 0 & -2 & -2\\ -2 & -5 & -8 & -4 & 3 & -1\\ 6 & 0 & -1 & 2 & -7 & -5\\ -1 & -1 & 1 & -1 & 2 & 1\end{bmatrix}\xrightarrow[\substack{R_{13}(2)\\ R_{14}(-6)\\ R_{15}(1)}]{R_{12}(-2)}\begin{bmatrix}1 & -2 & 3 & -1 & -1 & -2\\ 0 & 3 & -5 & 2 & 0 & 2\\ 0 & -9 & -2 & -6 & 1 & -5\\ 0 & 12 & -19 & 8 & -1 & 7\\ 0 & -3 & 4 & -2 & 1 & -1\end{bmatrix}$$

$$\xrightarrow[\substack{R_{24}(-4)\\ R_{25}(1)}]{R_{23}(3)}\begin{bmatrix}1 & -2 & 3 & -1 & -1 & -2\\ 0 & 3 & -5 & 2 & 0 & 2\\ 0 & 0 & -17 & 0 & 1 & 1\\ 0 & 0 & 1 & 0 & -1 & -1\\ 0 & 0 & -1 & 0 & 1 & 1\end{bmatrix}\xrightarrow{R_{34}}\begin{bmatrix}1 & -2 & 3 & -1 & -1 & -2\\ 0 & 3 & -5 & 2 & 0 & 2\\ 0 & 0 & 1 & 0 & -1 & -1\\ 0 & 0 & -17 & 0 & 1 & 1\\ 0 & 0 & -1 & 0 & 1 & 1\end{bmatrix}$$

$$\xrightarrow[R_{35}(1)]{R_{34}(1)}\begin{bmatrix}1&-2&3&-1&-1&-2\\0&3&-5&2&0&2\\0&0&1&0&-1&-1\\0&0&0&0&-16&-16\\0&0&0&0&0&0\end{bmatrix}$$

四列有效，$\text{rank}(A)=4$

6. 令 A 爲映射 T 代表矩陣 $T(\vec{x})=A\vec{x}$

$$A\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}x_1-x_2+2x_3\\2x_1+x_2\\-x_1-2x_2+2x_3\end{bmatrix}=\begin{bmatrix}1&-1&2\\2&1&0\\-1&-2&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix},\quad A=\begin{bmatrix}1&-1&2\\2&1&0\\-1&-2&2\end{bmatrix}$$

(1) $T(\vec{x}+\vec{y})=A(\vec{x}+\vec{y})=A\vec{x}+A\vec{y}=T(\vec{x})+T(\vec{y})$

$T(\alpha\vec{x})=A(\alpha\vec{x})=\alpha\cdot A(\vec{x})=\alpha\cdot T(\vec{x})$，$T$ is linear mapping

(2) $\forall\vec{x}\in N(A)$，$A\vec{x}=\vec{0}$

$$A\xrightarrow[R_{13}(1)]{R_{12}(-2)}\begin{bmatrix}1&-1&2\\0&3&-4\\0&-3&4\end{bmatrix}\xrightarrow[R_2\left(\frac{1}{3}\right)]{R_{23}(1)}\begin{bmatrix}1&-1&2\\0&1&\frac{-4}{3}\\0&0&0\end{bmatrix}$$

$$\xrightarrow{R_{21}(1)}\begin{bmatrix}1&0&\frac{2}{3}\\0&1&\frac{-4}{3}\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\vec{0},\quad \begin{matrix}x_3=3c,\ x_1=-2c\\x_2=4c\end{matrix}$$

$\vec{x}=c\begin{bmatrix}-2\\4\\3\end{bmatrix}$，$\left\{\begin{bmatrix}-2\\4\\3\end{bmatrix}\right\}$ 爲 $N(T)=N(A)$ 之基底

$\text{range}(T)=\text{col}(A)$，基底爲 $\left\{\begin{bmatrix}1\\2\\-1\end{bmatrix},\begin{bmatrix}-1\\1\\-2\end{bmatrix}\right\}$

當 $\begin{bmatrix}a\\b\\c\end{bmatrix}\in\text{range}(T)$，即 $\begin{bmatrix}a\\b\\c\end{bmatrix}=\alpha\begin{bmatrix}1\\2\\-1\end{bmatrix}+\beta\begin{bmatrix}-1\\1\\-2\end{bmatrix}$，$\alpha,\beta$ 爲任意數

$\text{rank}(T)=\text{rank}(A)=2$

(3)

$$\text{當} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = k \begin{bmatrix} -2 \\ 4 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in N(T), \quad \text{nullity}(T) = 1$$